

How to Strengthen the Security of RSA-OAEP

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Abstract—OAEP is one of the few standardized and widely deployed public-key encryption schemes. It was designed by Bellare and Rogaway as a scheme based on a trapdoor permutation such as RSA. RSA-OAEP is standardized in RSA’s PKCS #1 v2.1 and is part of several standards. OAEP was shown to be IND-CCA secure assuming the underlying trapdoor permutation is partial one-way, and RSA-OAEP was proven to be IND-CCA under the standard RSA assumption, both in the random oracle model. However, the latter reduction is not tight, meaning that the guaranteed level of security is not very high for a practical parameter choice. We observe that the situation is even worse because both analyses were done in the single-query setting, i.e. where an adversary gets a single challenge ciphertext. This does not take into account the fact that in reality an adversary can observe multiple ciphertexts of related messages. The results about the multi-query setting imply that the guaranteed concrete security can degrade by a factor of q , which is the number of challenge ciphertexts an adversary can get. We propose a very simple modification of the OAEP encryption, which asks that the trapdoor permutation instance is only applied to a part of the OAEP transform. We show that IND-CCA security of this scheme is tightly related to the hardness of one-wayness of the trapdoor permutation in the random oracle model. This implies tight security for RSA-OAEP under the RSA assumption. We also show that security does not degrade as the number of ciphertexts an adversary can see increases. Moreover, OAEP can be used to encrypt long messages without using hybrid encryption. We believe that this modification is easy to implement, and the benefits it provides deserves the attention of standard bodies.

Index Terms—Encryption, provable security, RSA-OAEP, cryptography standards.

I. INTRODUCTION

This paper combines the results of the following two papers: Kazukuni Kobara and Hideki Imai, “OAEP++ : A Very Simple Way to Apply OAEP to Deterministic OW-CPA Primitives,” *Cryptology ePrint Archive, Report 2002/130*, 2002, and Alexandra Boldyreva, “Strengthening Security of RSA-OAEP,” in *Proceedings of the Topics in Cryptology - CT-RSA 2009, The Cryptographers’ Track at the RSA Conference 2009*, LNCS, 2009. BACKGROUND AND MOTIVATION. OAEP

is one of the few standardized and widely deployed public-key encryption schemes. It was designed by Bellare and Rogaway [5] as a scheme based on a trapdoor permutation such as

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RSA. RSA-OAEP is standardized in RSA’s PKCS #1 v2.1 and is part of the ANSI X9.44, IEEE P1363, ISO 18033-2 and SET standards. The scheme is parameterized by k_0, k_1 . The encryption algorithm of $\text{OAEP}[F]$ takes a public key f , which is an instance of a trapdoor permutation family F , and a message M , picks k_0 -bit string r at random, pads M with k_1 zeros to get M' and computes the ciphertext $C = f(s || t)$ for $s = G(r) \oplus M'$ and $t = H(s) \oplus r$, where G and H are hash functions. $\text{OAEP}[F]$ was proven to be IND-CPA secure assuming F is a one-way trapdoor permutation family [5] and IND-CCA secure assuming F is partial one-way [12], both in the random oracle (RO) model, i.e., where G and H are modeled as random oracles [4]. Partial one-wayness is a stronger property than one-wayness and it asks that given the result of applying a random instance of the function family to a random point x it be hard to compute the first part of x . RSA is believed to be one-way, so under this assumption the result of [5] implies that $\text{OAEP}[\text{RSA}]$ (RSA-OAEP) is IND-CPA in the RO model. In [12] it was shown that one-wayness of RSA also implies partial one-wayness, therefore RSA-OAEP is IND-CCA under the standard RSA assumption (stating that RSA is one-way), in the RO model.

While the concrete security reduction showing OAEP is IND-CCA secure assuming partial one-wayness of the underlying permutation family is tight, the concrete bound showing RSA-OAEP is IND-CCA under the RSA assumption is quite loose, due to the “lossy” reduction from partial one-wayness to one-wayness of RSA. Such a loose concrete security bound implies that it may be easier to break the scheme than to invert RSA, and to maintain reasonable security guarantees one would need to use the scheme with a larger security parameter. It was shown in [17] that keys of length about 4-5 thousand bits are necessary, i.e. at least 4 times larger than the standard 1024-bit keys, and this means decryption will be about $64 = 4^3$ times slower than before (since decryption requires a modulo exponentiation whose complexity is cubic in the length of the security parameter). This is basically impractical.

Moreover, we note that the definitions of security of encryption in [5], [12] only consider an adversary given a single challenge ciphertext. In reality, of course, an adversary can observe multiple ciphertexts of possibly related messages. Such mismatch was studied in [3], [2], who defined security in the “multi-query” setting where the adversary can see multiple challenge ciphertexts on messages of its choice¹. The result of [3], [2] implies that security (IND-CPA or IND-CCA) in the single-query setting implies security in the multi-query setting, however, concrete security degrades as the number of

¹These works consider what they call a “multi-user” setting which also allows the adversary to see multiple challenge ciphertexts under multiple public keys. We do not consider multiple public keys in this work.

queries increases, and this loss cannot be avoided in general. However it is possible for some specific constructions, e.g. [3] shows that IND-CPA security of the ElGamal encryption scheme [11] stays tightly related to security of the decisional Diffie-Hellman problem regardless of how many queries an adversary makes. Concrete security in the multi-query setting of RSA-OAEP has not been explicitly addressed before our work. The general reduction in [3] implies that in the multi-query setting the IND-CCA security of RSA-OAEP is even worse by a factor of q , the number of challenge ciphertexts the adversary sees, than what was implied by already loose bound from [12].

OUR CONTRIBUTIONS. We propose an extremely simple modification to the RSA-OAEP scheme that permits several security improvements. Unlike most of alternative constructions that have been suggested [18], [9], [16], the modification we suggest does not change the transform construction. The modified scheme, which we called OAEP++ in [14], differs from OAEP in that it uses trapdoor permutations of particular structure. Informally, they just leave the last part of the input (t -part of the output of the OAEP transform) in the clear. The scheme can be immediately instantiated with the RSA family if we apply an RSA function only to the s -part of the OAEP transform output, or to a portion of the s -part.

We show that RSA-OAEP++ is IND-CCA secure in the RO model under the standard RSA assumption and the reduction is tight. Moreover, not only the bound in the reduction is significantly improved, but also the running time of the adversary in the reduction.

It turns out that this simple modification has even more advantages. We prove that concrete IND-CCA security of the modified RSA-OAEP scheme stays tightly related to onewayness of RSA regardless of how many challenge ciphertexts an adversary sees (is independent of parameter q). The proof uses the self-reducibility property of RSA, and it does not seem to be possible to apply the same idea to the original RSA-OAEP scheme. Hence, the modified RSA-OAEP provides significantly better security guarantees than the original version, for very practical parameter sizes, which results in a very efficient scheme.

Additionally, the modified RSA-OAEP scheme can be used to encrypt long messages without using symmetric encryption in the hybrid encryption construct. For that the function G in the transform is made variable-output-length, i.e. its output size is of the length of the message plus the zero padding of length k_1 . For a fixed-output-length hash $G'(\cdot)$ one can efficiently construct $G(\cdot)$ as $G'(\langle 0 \rangle \| \cdot) \| G'(\langle 1 \rangle \| \cdot) \dots \| G'(\langle l \rangle \| \cdot)$, where $\langle i \rangle$ means the binary representation of the counter $i \in \mathbb{N}$. The function H in the transform needs to be variable-input-length, which is not a problem. The RSA function is applied to the first k (e.g. 1024) bits of the s -part of the OAEP transform. The proof of security stays virtually the same. This scheme yields more compact ciphertexts for long messages than the one obtained through the use of hybrid encryption because there is no need to encrypt the symmetric key.

Finally, we show that a further small modification of the RSA-OAEP++ helps to eliminate a lesser-known minor weak-

ness of RSA-OAEP. Namely, the proofs of security [5], [12] ignore the fact that the output of the transform is random in \mathbb{Z}_k , when it needs to be random in the domain of the permutation instance f (\mathbb{Z}_N^* in the case of RSA, where N is $2^{k-1} \leq N < 2^k$). It is pointed out in [12] that this discrepancy can be resolved by making the encryption algorithm repeatedly compute the transform with fresh random coins until the result is in \mathbb{Z}_N^* . While practical, this solution makes the encryption and the reduction expected polynomial-time. We show how to eliminate this problem for RSA-OAEP++ without the need to use fresh random coins multiple times.

We hope the standard bodies will pay attention to the modified RSA-OAEP as the advantages it offers seem to be well worth a very simple modification to the standard scheme.

MORE RELATED WORK. After it was realized by [12] that IND-CCA security of RSA-OAEP is not tight there appeared several alternative encryption schemes using different transforms before applying the RSA function. These include OAEP+ [18], SAEP+ [9], REACT [16]. Another alternative, which was proposed in [19] is the simplest construction and is known as Simple RSA or RSA-KEM; it can be viewed as a special case of REACT. OAEP+ is slightly less efficient than OAEP, SAEP+ and OAEP++ as it uses 3 hash function applications. Compared to OAEP++, OAEP+ and SAEP+ have ciphertexts of length of RSA modulus. REACT and RSA-KEM are encryption schemes suitable for encryption of long messages. OAEP++ yields more compact ciphertexts for long messages than REACT and RSA-KEM. IND-CCA security of all of these schemes are tightly related to the hardness of the RSA problem, in the RO model and in the single-query setting. The latter two schemes (REACT and RSA-KEM), unlike the former two, can also be shown to have an improved security reduction in the multi-query setting (though it was not formally proven). We think it is important to show that the standardized RSA-OAEP scheme has similar properties, with the help of a very simple modification that should be easy to implement, because it appears very hard to replace the standardized schemes with completely different constructions.

Improving the concrete security bounds is very important. Many papers besides the aforementioned work of [3] focused on this issue. For example, Coron [10] showed a new proof with improved security reduction for the RSA-based Full-Domain Hash signature scheme and his technique has been widely used since then. Abe et al. [1] improved the time bound in the security proofs of some of RSA-based encryption schemes by considering 4-round Feistel network transformation.

It was shown in [15] that the OAEP++ transform is universal in that it can also serve as a padding for signing with trapdoor permutations. OAEP++ has been also used in [8] for an orthogonal reason of showing some positive results about non-malleability of OAEP when one or both ROs are instantiated with existing functions, and those results serve as an additional motivation for OAEP++. The paper [8] neither considers the multi-query setting nor provides concrete security bounds.

II. PRELIMINARIES

NOTATION AND CONVENTIONS. We denote by $\{0, 1\}^*$ the set of all binary strings of finite length. We will refer to members of $\{0, 1\}^*$ as strings. If X, Y are strings then $X\|Y$ denotes the concatenation of X and Y . If S is a set then $X \stackrel{\$}{\leftarrow} S$ denotes that X is selected uniformly at random from S . If $k \in \mathbb{N}$ then 1^k denotes the string consisting of k consecutive “1” bits. If A is a randomized algorithm and $n \in \mathbb{N}$, then the notation $X \stackrel{\$}{\leftarrow} A(X_1, X_2, \dots, X_n)$ denotes that X is assigned the outcome of the experiment of running A on inputs X_1, X_2, \dots, X_n . When describing algorithms, if X is a variable and Y is a string, then $X \leftarrow Y$ denotes that X is assigned the value of Y .

All algorithms we consider are possibly randomized unless indicated otherwise. By convention, the running-time of an algorithm is measured relative to the bit-length of the input and refers to both the actual running-time and program size, including that of any overlying experiment, according to some fixed RAM model of computation. k denotes the security parameter. All algorithms we consider run in time polynomial (or expected polynomial) in k .

SYNTAX OF PUBLIC-KEY ENCRYPTION. A public-key encryption (PKE) scheme $\mathcal{PE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ with associated message space MsgSp , which may depend on the security parameter k , consists of three algorithms. The key-generation algorithm \mathcal{K} on input 1^k returns a public key pk and matching secret key sk . The encryption algorithm \mathcal{E} takes pk and a plaintext M to return a ciphertext. The deterministic decryption algorithm \mathcal{D} takes sk and a ciphertext C to return a plaintext. The consistency condition requires that for all $k \in \mathbb{N}$ and all $M \in \text{MsgSp}(k)$ the probability of $\mathcal{D}_{sk}(C) = M$ is 1, where the probability is over the experiment

$$(pk, sk) \stackrel{\$}{\leftarrow} \mathcal{K}(1^k); C \stackrel{\$}{\leftarrow} \mathcal{E}_{pk}(M).$$

SECURITY OF PKE. We recall the notions of security of public-key encryption (PKE). We only consider the definitions addressing chosen-ciphertext attack (as opposed to a weaker version for chosen-plaintext attack). We present two variants of the standard IND-CCA definition. In the first one the adversary is given a single challenge ciphertext, and in the second definition the adversary can see multiple challenge ciphertexts. We then show the relation between the definitions.

Definition 2.1: [Single- and Multi-query CCA Security of PKE] Let $\mathcal{PE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be a PKE scheme. Let the left or right selector be the map LR defined by $\text{LR}(M_0, M_1, b) = M_b$ for all equal-length strings M_0, M_1 , and for any $b \in \{0, 1\}$. For an adversary A define the experiment:

Experiment $\text{Exp}_{\mathcal{PE}, A}^{\text{ind-cca}}(1^k)$

$$b \stackrel{\$}{\leftarrow} \{0, 1\}$$

$$(pk, sk) \stackrel{\$}{\leftarrow} \mathcal{K}(1^k)$$

$$d \stackrel{\$}{\leftarrow} A^{\mathcal{E}_{pk}(\text{LR}(\cdot, \cdot, b)), \mathcal{D}_{sk}(\cdot)}$$

If $b = d$ then return 1 else return 0

It is mandated the LR encryption oracle (also known as the challenge oracle) is queried on pairs of messages in $\text{MsgSp}(k)$ and of equal length and the decryption oracle is not queried on the outputs of the LR encryption oracle.

For an adversary A who is allowed to make a single query to its challenge oracle (we will refer to such an adversary a single-query adversary) define the single-query(cca)-advantage, $\text{Adv}_{\mathcal{PE}, A}^{\text{ind-cca-sq}}(k)$ as

$$2 \cdot \Pr \left[\text{Exp}_{\mathcal{PE}, A}^{\text{ind-cca}}(1^k) = 1 \right] - 1.$$

We define the multi-query(mq)-cca-advantage, $\text{Adv}_{\mathcal{PE}, A}^{\text{ind-cca-mq}}(k)$ the exact same way, but for the adversary A who can query its challenge oracle an arbitrary number of times. We will refer to such A a multi-query adversary.

A scheme \mathcal{PE} is said to be IND-CCA secure in the single- (resp. multi-) query setting if the single-query (resp. multi-query) -cca-advantage of any polynomial-time adversary is negligible. ■

It is shown by using a hybrid argument in [3], [2] that for any $k \in \mathbb{N}$, a scheme \mathcal{PE} and any multi-query adversary A making q queries to its challenge oracle there exists a single-query adversary B so that

$$\text{Adv}_{\mathcal{PE}, A}^{\text{ind-cca-mq}}(k) \leq q \cdot \text{Adv}_{\mathcal{PE}, B}^{\text{ind-cca-sq}}(k), \quad (1)$$

where the running time of B is that of A plus $O(\log q)$, and B does the same number of decryption oracle queries as A .

It was also shown in [3] that the above bound is tight and cannot be improved in general. But for specific schemes, e.g. ElGamal, the concrete security in the multi-query setting is basically the same as in the single-query setting.

In this paper we are interested in improving the bound in concrete security treatment of the popular RSA-OAEP scheme in the multi-query setting. Accordingly we recall the computational assumptions used in the analyses of the scheme.

COMPUTATIONAL ASSUMPTIONS. A trapdoor-permutation generator is an algorithm \mathcal{F} that on input 1^k returns the description of a permutation and its inverse f, f^{-1} . The trapdoor property means that for every instance f there exist a function f^{-1} with the same domain and range so that $f(f^{-1}) \equiv f^{-1}(f) \equiv \text{ID}$, the identity function.

Definition 2.2: [One-wayness] A trapdoor permutation generator \mathcal{F} is called one-way if for every $k \in \mathbb{N}$ and every adversary I its advantage $\text{Adv}_{\mathcal{F}, I}^{\text{owf}}(k)$ defined as

$$\Pr \left[x = x' : x' \stackrel{\$}{\leftarrow} I(1^k, f, f(x)) \right],$$

where f, f^{-1} are generated by $\mathcal{F}(1^k)$ and x is picked at random from the domain D of every f output by \mathcal{F} , is negligible. ■

Definition 2.3: [Partial-Domain One-wayness] A trapdoor permutation generator \mathcal{F} is called partial-domain one-way for $k \in \mathbb{N}$ and some extra parameter $k' \leq k$, which can be a linear function of k , if for every $k \in \mathbb{N}$ and every adversary I its advantage $\text{Adv}_{\mathcal{F}, I}^{\text{pd-owf}}(k, k')$ defined as

$$\Pr \left[x[1 \dots k'] = x' : x' \stackrel{\$}{\leftarrow} I(1^k, f, f(x)) \right],$$

where f, f^{-1} are generated by $\mathcal{F}(1^k)$, x is picked at random from the domain D of every f output by \mathcal{F} and $x[1 \dots k']$ denotes the first k' bits of x , is negligible. ■

An RSA trapdoor permutation generator is an algorithm that on input 1^k returns $(N, e), (N, d)$ where N is the product of two random distinct $\lfloor k/2 \rfloor$ -bit primes and $ed \equiv 1 \pmod{\phi(N)}$. (Here $\phi(\cdot)$ is Euler's phi function.) The domain and range here are \mathbb{Z}_N^* .

The standard assumption is that the RSA trapdoor permutation generator is one-way, and the reasonable security level requires k to be at least 1024 bits. It was shown in [12] that under this assumption RSA is also partial one-way. But the concrete reduction in [12] is not tight showing that a much larger RSA modulus is required to guarantee reasonable level of the stronger notion of partial one-wayness.

III. OAEP AND ITS SECURITY

OAEP ENCRYPTION. The OAEP encryption [5] is parameterized by k_0, k_1 and k_2 (that can be linear functions of k , but typically $k_0 = k_1 = 128$ and $k_2 = k$) and makes use of a trapdoor permutation generator \mathcal{F} with domain and range $\{0, 1\}^{k_2}$ and two random oracles

$$\begin{aligned} G &: \{0, 1\}^{k_0} \rightarrow \{0, 1\}^{k_2 - k_0} \quad \text{and} \\ H &: \{0, 1\}^{k_2 - k_0} \rightarrow \{0, 1\}^{k_0}. \end{aligned}$$

The message space is $\{0, 1\}^{k_2 - k_0 - k_1}$. The scheme $\text{OAEP}[\mathcal{F}] = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ is defined as follows:

- The key generation algorithm $\mathcal{K}(1^k)$ picks a pair $(f, f^{-1}) \xleftarrow{\$} \mathcal{F}(1^{k_2})$ and returns f as pk and f^{-1} as sk .
- The encryption algorithm $\mathcal{E}(pk, M)$ picks $r \xleftarrow{\$} \{0, 1\}^{k_0}$, computes $s \leftarrow G(r) \oplus (M \parallel 0^{k_1})$, $t \leftarrow H(s) \oplus r$ and $C \leftarrow f(s \parallel t)$ and returns C .
- The decryption algorithm $\mathcal{D}(sk, C)$ computes $s \parallel t \leftarrow f^{-1}(C)$, $r \leftarrow t \oplus H(s)$ and $M \leftarrow s \oplus G(r)$. If the last k_1 bits of M are zeros, then it returns the first $k_2 - k_0 - k_1$ bits of M , otherwise it returns \perp .

SECURITY OF OAEP. The encryption scheme $\text{OAEP}[\mathcal{F}]$ is IND-CCA secure in the RO model if the underlying trapdoor permutation generator \mathcal{F} is partial-domain one-way [12]. The concrete security results in [12] are done for the single-query IND-CCA security. We “translate” them into the multi-query IND-CCA security using the result from [3] recalled in Equation 1.

Theorem 3.1: [12], [3] Let \mathcal{F} be a trapdoor permutation generator with domain and range $\{0, 1\}^k$. Let $\text{OAEP}[\mathcal{F}]$ be the encryption scheme defined above. Then for any adversary A making q_e challenge oracle and q_d decryption oracle queries, q_H, q_G queries to RO oracles H and G , there exists an adversary B such that

$$\begin{aligned} \text{Adv}_{\mathcal{F}, B}^{\text{pd-owf}}(k, k_2 - k_0) &\geq \frac{\text{Adv}_{\text{OAEP}[\mathcal{F}], A}^{\text{ind-cca-mq}}(k)}{2q_e q_H} \\ &\quad - \frac{1}{q_e q_H} \left(\frac{q_d q_G + q_d + q_G}{2^{k_0}} + \frac{q_d}{2^{k_1}} \right), \end{aligned}$$

and the running time of B is that of A plus $q_G \cdot q_H \cdot (T_{\mathcal{F}}(k) + O(1)) + O(\log q_e)$, where $T_{\mathcal{F}}(k)$ is the time needed for evaluating a random instance of \mathcal{F} . ■

As we can see the reduction is not particularly tight, but the situation becomes even worse if we use RSA, pretty much the only practical trapdoor permutation. It is believed to be one-way, and it was shown in [12] that under this assumption it is partial one-way as well, but the reduction is not tight. The concrete result is as follows.

Theorem 3.2: [12], [3] Consider the RSA trapdoor permutation generator with domain and range $\{0, 1\}^k$. Let $\text{OAEP}[\text{RSA}]$ be the encryption scheme defined above². Then for any adversary A making q_e challenge oracle and q_d decryption oracle queries, q_H, q_G queries to RO oracles H and G there exists an adversary B such that

$$\begin{aligned} \text{Adv}_{\text{RSA}, B}^{\text{owf}}(k) &\geq \frac{(\text{Adv}_{\text{OAEP}[\text{RSA}], A}^{\text{ind-cca-mq}}(k))^2}{4q_e} \\ &\quad - \frac{1}{q_e} \text{Adv}_{\text{OAEP}[\text{RSA}], A}^{\text{ind-cca-mq}}(k) \left(\frac{q_d q_G + q_d + q_G}{2^{k_0}} \right. \\ &\quad \left. + \frac{q_d}{2^{k_1}} + \frac{32}{2^{k-2k_0}} \right), \end{aligned}$$

and the running time of B is 2 times that of A plus $q_H \cdot (q_H + 2q_G) \cdot O(k^3) + O(\log q_e)$. ■

Such a loose concrete security bound implies that to maintain reasonable security guarantees, i.e. so that it not much harder to break the scheme than to invert 1024-bit RSA, one would need to use the scheme with a larger security parameter. It is shown in [17] that keys of length about 4-5 thousand bits are necessary, i.e. at least 4 times larger than the standard 1024-bit keys, and this means decryption will be about $64 = 4^3$ times slower than before (since decryption requires a modulo exponentiation whose complexity is cubic in the length of the parameters). This is basically impractical. Note that this estimate is for $q_e = 1$, i.e. when a single challenge ciphertext is considered. If we take into account the maximum number of queries to the challenge oracle an adversary makes $- q_e$, then to have reasonable security guarantees in the practical multi-query settings the RSA parameters should be even larger, making the scheme's algorithms prohibitively slow.

IV. SIMPLE MODIFICATION – OAEP++, AND ITS SECURITY

We propose an extremely simple modification, which permits several concrete security improvements. The modified scheme differs from $\text{OAEP}[\mathcal{F}]$ in that it uses trapdoor permutations of particular structure, which leave the last part of the input in the clear. Let \mathcal{F} be a generator producing trapdoor permutations with domain and range $\{0, 1\}^k$. Define a new generator \mathcal{F}_k first to run \mathcal{F} ; let (f, f^{-1}) be its output of \mathcal{F} , and define the first output of \mathcal{F}_k as $f_p(x) \equiv f(x[1, \dots, k]) \parallel \text{ID}(x[k+1, \dots, p]) = f(x[1, \dots, k]) \parallel x[k+1, \dots, p]$ for any inputs x of length $p \geq k$, where $x[1, \dots, k]$ denotes the first k bits of x . The second output, the inverse

²We comment on the mismatch between the domains \mathbb{Z}_N^* and $\{0, 1\}^k$ in Section VII.

permutation, is defined straight-forwardly. With regard to the OAEP construction we will be interested in cases when $p = k_2$ and $k \leq k_2 - k_0$, so that applying \mathcal{F}_k to the output of the OAEP transform leaves the t -part in the clear.

It is not hard to see that if \mathcal{F} is one-way, then \mathcal{F}_k is partial one-way, in that it is infeasible to recover first k bits of the preimage. With respect to RSA, we get that RSA_k , applying RSA to only the first k bits of the input, is partial-one-way under the standard RSA assumption. That immediately implies that $\text{OAEP}[\mathcal{F}_k]$, when $k \leq k_2 - k_0$ is IND-CCA in the RO model, if \mathcal{F} is one-way, and we get that $\text{OAEP}[\text{RSA}_k]$ is IND-CCA in the RO model under the standard RSA assumption³. For the concrete security result we can use the bound of Theorem 3.1.

But we can do even better. First, we can get rid of factor q_h . This is possible for the modified scheme for the following reason. The proof of the original scheme constructs an adversary B breaking partial one-wayness of \mathcal{F} using the IND-CCA adversary A for $\text{OAEP}[\mathcal{F}]$. B needs to partially invert its input $y = f(s || t)$, i.e. find s . This input y is given to A as the challenge ciphertext. The proof argues that the only way A can win the IND-CCA game is by querying the random oracle H on s at some point. While B cannot check which of the RO queries A made is the correct value B is looking for (since B does not know the second part t to verify this), it can just pick one query at random. This is where the factor q_h , the number of RO queries, is coming from. For the modified scheme, the proof from [12] applies without a single change, except we can note that B will now be able to select the correct s out of A 's RO queries because t is in the clear. B just checks if $f(s_i || t) = y$ for all queries to the random oracle H that A makes.

In addition, the running time of the constructed adversary in the reduction for the modified scheme is also improved. We comment on this below and give more details in the next section.

Here is the security result for the modified scheme used with any trapdoor permutation generator.

Theorem 4.1: [14], [3] Let \mathcal{F} be a trapdoor permutation generator with domain and range $\{0, 1\}^k$. Let \mathcal{F}_k be a trapdoor permutation generator producing permutations with domain and range $\{0, 1\}^p$ for $p \geq k$ as defined above. Let $\text{OAEP}[\mathcal{F}_k] = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the encryption scheme defined in Section III so that $p = k_2$ and $k \leq k_2 - k_0$. Then for any adversary A making q_e challenge oracle queries, q_d decryption oracle queries, q_H, q_G queries to RO oracles H and G there exists an adversary B such that

$$\begin{aligned} \text{Adv}_{\mathcal{F}, B}^{\text{owf}}(k) &\geq \frac{\text{Adv}_{\text{OAEP}[\mathcal{F}_k], A}^{\text{ind-cca-mq}}(k)}{2q_e} \\ &\quad - \frac{1}{q_e} \left(\frac{q_d q_G + q_d + q_G}{2^{k_0}} + \frac{q_d}{2^{k_1}} \right), \end{aligned}$$

and the running time of B is that of A plus $q_d \cdot q_H \cdot (T_{\mathcal{F}_k}(k) + O(1))$, where $T_{\mathcal{F}_k}(k)$ is the time needed for evaluating a random instance of \mathcal{F}_k . ■

³This was previously observed in [8]

Note an improvement compared to the running time of the adversary in the proof of Theorem 3.1. The number of trapdoor permutation computations there is proportional to $q_G \cdot q_H$. Here it is proportional to $q_d \cdot q_H$. This is much better as in practice the number of decrypted ciphertexts can be much smaller than the number of hash computations. We explain the reason for this saving in the next section, where we provide the proof.

The RSA instantiation result is immediate if we use an RSA trapdoor permutation generator in place of \mathcal{F} and RSA_k in place of \mathcal{F}_k above.

V. IMPROVING THE SECURITY IN THE MULTI-QUERY SETTING

For the RSA instantiation of the proposed scheme we can get rid of the factor q_e in the bound of Theorem 4.1. This shows that security in the multi-query setting does not have to degrade as more messages are encrypted by each user (when an adversary does multiple queries to the challenge encryption oracle). Hence, the modified scheme provides even better security guarantees than the original version, for very practical parameter sizes. The following theorem states the improvement result.

Theorem 5.1: Let RSA be a trapdoor permutation generator with domain and range $\{0, 1\}^k$. Let RSA_k be a trapdoor permutation generator with domain and range $\{0, 1\}^p$ for $p \geq k$ as defined in Section IV. Let $\text{OAEP}[\text{RSA}_k]$ be the encryption scheme defined in Section III so that $k_2 = p$ and $k \leq k_2 - k_0$. Then for any adversary A attacking IND-CCA security of the scheme making at most q_e queries to its challenge oracle, q_d decryption oracle queries, q_H, q_G queries to RO oracles H and G , there exists an adversary B such that

$$\begin{aligned} \text{Adv}_{\text{RSA}, B}^{\text{owf}}(k) &\geq \frac{\text{Adv}_{\text{OAEP}[\text{RSA}_k], A}^{\text{ind-cca-mq}}(k)}{2} \\ &\quad - \left(\frac{q_d q_G + q_e q_d + q_e q_G}{2^{k_0}} + \frac{q_d}{2^{k_1}} \right), \end{aligned}$$

and the running time of B is that of A plus $(q_e + 1) \cdot T_m(k) + (q_d \cdot q_H + 1) \cdot T_e(k) + O(\log q_e)$, where $T_m(k)$ and $T_e(k)$ are times required to compute one modulo multiplication and exponentiation respectively in \mathbb{Z}_N^* , where $N < 2^k$. ■

What does the improvement mean in practice? The current belief is that 1024-bit RSA provides 80 bits of security, so for any adversary B with reasonable resources $\text{Adv}_{\text{RSA}, B}^{\text{owf}}(k) \leq 2^{-80}$ (and there are indications that this estimate is outdated in that it does not take into account newer attacks and growing computing power, and the bound is likely to be higher [13]). Now assume an adversary manages to obtain 2^{20} ciphertexts of chosen messages. This is about the number of TLS connections that were required to mount the well-known attack on RSA-PKCS1 by Bleichenbacher [7] (though his attacks needed that many chosen ciphertexts). Then according to Theorem 4.1 the bound on $\text{Adv}_{\text{OAEP}[\text{RSA}_k], A}^{\text{ind-cca-mq}}(k)$ is only about 2^{-59} , which is not a strong security level. Theorem 5.1 implies that in fact security of the scheme does not degrade as an adversary mounts more chosen-plaintext attacks and stays tightly related to the assumed security level of the underlying RSA problem.

We now give a single proof for the more general Theorem 4.1 and Theorem 5.1 dealing with the RSA-based instantiation.

PROOF FOR THEOREMS 4.1 AND 5.1. We show how to modify the proof of security from [12], which assumes an adversary A attacking IND-CCA security of OAEP[\mathcal{F}] in the single-query setting (in the RO model). In our case we consider a special case of the scheme, OAEP[\mathcal{F}_k] and the practical instantiation OAEP[RSA_k] (i.e. when the permutation is applied to the first k bits of the OAEP output, leaving the t -part in the clear).

Following [12] we use the game-playing technique of [6], [20] and consider a sequence of experiments or games, associated with the adversary A and random oracles $G(\cdot), H(\cdot)$. For the most part the proof is a simple extension of the proof in [12]. For $i \in \mathbb{N}$ we let $\Pr[\text{Game}_i]$ denote the probability that Game i outputs 1.

Game-0 $^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r , return $G(r)$
 when $H(\cdot)$ is queried on s , return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Pick $r_i^* \xleftarrow{\$} \{0, 1\}^{k_0}$
 Compute $s_i^* \leftarrow G(r_i^*) \oplus M_{b,i} \| 0^{k_1}$
 Compute $C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \leftarrow H(s_i^*) \oplus r_i^*$
 return $C_i^* \| t_i^*$
 when A makes j -th query $C_j \| t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \| t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$

Game 0 corresponds to $\text{Exp}_{\text{OAEP}[\mathcal{F}_k], A}^{\text{ind-cca}}(1^k)$, the standard multi-query IND-CCA experiment (c.f. Definition 2.1 for the multi-query adversary). Each of q_e challenge ciphertexts is generated according to the definition of encryption of OAEP[RSA_k] as follows. For $1 \leq i \leq q_e$, to encrypt $M_{i,b}$ first r_i^* is chosen at random from $\{0, 1\}^{k_0}$. Then $C_i \leftarrow f_k(s_i^* \| t_i)$, where $s_i^* = G(r_i^*) \oplus M_{i,b} \| 0^{k_1}$ and $t_i = r_i^* \oplus H(s_i^*)$. Decryption oracle queries are answered according to the decryption algorithm of OAEP[\mathcal{F}_k]. By construction and Definition 2.1 we get

$$\frac{1}{2} + \frac{1}{2} \cdot \text{Adv}_{\text{OAEP}[\mathcal{F}_k], A}^{\text{ind-cca-mq}}(k) = \Pr[\text{Game}_0].$$

Game 1 is different from Game 0 in that it moves the computation of the random coins, $r_1^+, \dots, r_{q_e}^+$, used in the challenge ciphertexts explicitly up front, together with the computations of $g_1^+, \dots, g_{q_e}^+$, the values simulating the corresponding random oracle G answers. By computation we mean choosing the values at random from the corresponding domains ($\{0, 1\}^{k_0}$ and $\{0, 1\}^{k_2-k_0}$ resp.) and storing the results. Further in the game r_i^+ is used in place of r_i^*

Game-1 $^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 For $1 \leq i \leq q_e$ pick
 $r_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}, g_i^+ \xleftarrow{\$} \{0, 1\}^{k_2-k_0}$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 and $r = r_l^+$ for some $1 \leq l \leq q_e$
 then return g_l^+ , otherwise return $G(r)$
 when $H(\cdot)$ is queried on s , return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $s_i^* \leftarrow g_i^+ \oplus M_{b,i} \| 0^{k_1}$
 Compute $C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \leftarrow H(s_i^*) \oplus r_i^+$
 return $C_i^* \| t_i^*$
 when A makes j -th query $C_j \| t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \| t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$

Game-2 $^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 For $1 \leq i \leq q_e$ pick
 $r_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}, g_i^+ \xleftarrow{\$} \{0, 1\}^{k_2-k_0}$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 and $r = r_l^+$ for some $1 \leq l \leq q_e$
 then return $G(r)$, otherwise return $G(r)$
 when $H(\cdot)$ is queried on s , return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $s_i^* \leftarrow g_i^+ \oplus M_{b,i} \| 0^{k_1}$
 Compute $C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \leftarrow H(s_i^*) \oplus r_i^+$
 return $C_i^* \| t_i^*$
 when A makes j -th query $C_j \| t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \| t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$

and g_i^+ is used in place of $G(r_i^+)$, for all $1 \leq i \leq q_e$. I.e. each challenge ciphertext has the form $f_k(s_i^* \| t_i^*)$, where $s_i^* = (M_{i,b} \| 0^{k_1}) \oplus g_i^+$, $t_i^* = h_i^+ \oplus r_i^*$ for $r_i^* = r_i^+$ and $h_i^+ = H(s_i^*)$. And whenever A queries the random oracle G on r_i^+ for any $1 \leq i \leq q_e$, it is given back g_i^+ . These changes do not affect the distribution of the view of A compared to that in Game 0, because $(r_1^+, G(r_1^+), \dots, r_{q_e}^+, G(r_{q_e}^+))$ and $(r_1^+, g_1^+, \dots, r_{q_e}^+, g_{q_e}^+)$ have the same distribution, since G is a random oracle:

$$\Pr[\text{Game}_1] = \Pr[\text{Game}_0].$$

Game-3 $_A^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 For $1 \leq i \leq q_e$ pick
 $r_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}, g_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}$
 $s_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}, h_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$
 when $H(\cdot)$ is queried on s
 and $s = s_l^+$ for some $1 \leq l \leq q_e$
 then return h_l^+ , otherwise return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $C_i^* \leftarrow f_k(s_i^+)$
 Compute $t_i^* \leftarrow h_i^+ \oplus r_i^+$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \parallel t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$

Game-4 $_A^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 For $1 \leq i \leq q_e$ pick
 $r_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}, g_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}$
 $s_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}, h_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$
 when $H(\cdot)$ is queried on s
 and $s = s_l^+$ for some $1 \leq l \leq q_e$
 then return $H(s)$, otherwise return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $C_i^* \leftarrow f_k(s_i^+)$
 Compute $t_i^* \leftarrow h_i^+ \oplus r_i^+$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \parallel t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$

Game 2 differs from Game 1 only in that the queries to the random oracle G on points $r_1^+, \dots, r_{q_e}^+$ made by the adversary or by the decryption oracle are answered at random independently from the values $g_1^+, \dots, g_{q_e}^+$ used to compute the challenge ciphertexts (e.g. by calling $G(\cdot)$). Hence the challenge ciphertexts are independent from the challenge bit b (since they are uniformly distributed, independent of the rest of A 's view) and

$$\Pr[\text{Game}_2] = \frac{1}{2}.$$

Game-5 $_A^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 For $1 \leq i \leq q_e$ pick
 $r_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}, g_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}$
 $s_i^+ \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}, h_i^+ \xleftarrow{\$} \{0, 1\}^{k_0}$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$
 when $H(\cdot)$ is queried on s
 return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $s_i^* \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}; C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \xleftarrow{\$} \{0, 1\}^{k_0}$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \parallel t_j)$
 Until A returns a bit d
 Return 1 iff $b = d$
 Game-6 $_A^{G(\cdot), H(\cdot)}(k)$:
 $b \xleftarrow{\$} \{0, 1\}; (f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$
 when $H(\cdot)$ is queried on s
 return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, ($1 \leq i \leq q_e$)
 Compute $s_i^* \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}; C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \xleftarrow{\$} \{0, 1\}^{k_0}$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, ($1 \leq j \leq q_d$)
 If $H(f_k^{-1}(C_j)) \oplus t_j$ was queried to $G(\cdot)$
 then return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \parallel t_j)$
 otherwise return \perp
 Until A returns a bit d
 Return 1 iff $b = d$

Similarly to [12] we can argue that the view of A and thus its outputs have the same distribution in Games 1 and 2 unless A queries G oracle on either of the points $r_1^*, \dots, r_{q_e}^*$ (directly or making the decryption oracle make this query). Let us denote the probability of such event in this game $\Pr[\text{AskG}_2]$, and such an event is defined similarly in the following games.

$$\Pr[\text{Game}_1] - \Pr[\text{Game}_2] \leq \Pr[\text{AskG}_2].$$

Game 3 is different from Game 2 in that it moves the computation of $s_1^+, \dots, s_{q_e}^+$ and $h_1^+, \dots, h_{q_e}^+$ explicitly up front. By computation we mean choosing the values at random from the corresponding domains ($\{0, 1\}^{k_2 - k_0}$ and $\{0, 1\}^{k_0}$ resp.) and storing the results. Further s_i^+ is used in place of s_i^* and h_i^+ is used in place of $H(s_i^+)$, for all $1 \leq i \leq q_e$. I.e. each challenge ciphertext has the form $f_k(s_i^* \parallel t_i^*)$, where

Game-7 $_A^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}$; $(f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$
 when $H(\cdot)$ is queried on s
 return $H(s)$
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, $(1 \leq i \leq q_e)$
 Compute $s_i^* \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}$; $C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \xleftarrow{\$} \{0, 1\}^{k_0}$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, $(1 \leq j \leq q_d)$
 If $f^{-1}(C_j)$ was queried to $H(\cdot)$
 and $H(f^{-1}(C_j)) \oplus t_j$ was queried to $G(\cdot)$
 then return $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(C_j \parallel t_j)$
 otherwise return \perp
 Until A returns a bit d
 Return 1 iff $b = d$

Game-8 $_A^{G(\cdot), H(\cdot)}(k)$:

$b \xleftarrow{\$} \{0, 1\}$; $(f_k, f_k^{-1}) \xleftarrow{\$} \mathcal{K}(1^k)$
 Run A on input f_k and
 when $G(\cdot)$ is queried on r
 then return $G(r)$; store $(r, G(r))$ in array G-list
 when $H(\cdot)$ is queried on s
 return $H(s)$; store $(s, H(s))$ in array H-list
 when A makes i -th query $(M_{0,i}, M_{1,i})$
 to $\mathcal{E}_{f_k}^{G(\cdot), H(\cdot)}(\text{LR}(\cdot, \cdot, b))$, $(1 \leq i \leq q_e)$
 Compute $s_i^* \xleftarrow{\$} \{0, 1\}^{k_2 - k_0}$; $C_i^* \leftarrow f_k(s_i^*)$
 Compute $t_i^* \xleftarrow{\$} \{0, 1\}^{k_0}$
 return $C_i^* \parallel t_i^*$
 when A makes j -th query $C_j \parallel t_j$
 to $\mathcal{D}_{f_k^{-1}}^{G(\cdot), H(\cdot)}(\cdot)$, $(1 \leq j \leq q_d)$
 If $f_k^{-1}(C_j)$ was queried to $H(\cdot)$ and $H(f_k^{-1}(C_j)) \oplus t_j$
 was queried to $G(\cdot)$,
 If G-list contains (r, G_r) and H-list contains (s, H_s) ,
 so that $H_s \oplus t_j = r$, $f_k(s \parallel t_j) = C_j \parallel t_j$
 and the last k_1 bits of $s \oplus G_r$ are zeros
 then return the rest of $s \oplus G_r$, otherwise return \perp
 Until A returns a bit d
 Return 1 iff $b = d$

$s_i^* = s_i^+$, $t_i^* = h_i^+ \oplus r_i^*$ for $r_i^* = r_i^+$ and $h_i^+ = H(s_i^*)$.
 And whenever A queries the random oracle H on s_i^+ for any
 $1 \leq i \leq q_e$, it is given back h_i^+ . These changes do not affect
 the distribution of the view of A compared to that in Game 2,
 because we replaced each quadruple $(s_i^*, H(s_i^*), g_i^+, b)$ with
 another having the same distribution, since H is a random
 oracle:

$$\Pr[\text{AskG}_2] = \Pr[\text{AskG}_3].$$

In Game 4, the difference with Game 3 is only in that the
 queries to the random oracle H made by the adversary or by

the decryption oracle on points $s_1^+, \dots, s_{q_e}^+$ are answered at
 random independently from the values $h_1^+, \dots, h_{q_e}^+$ used in
 the challenge ciphertexts (e.g by calling $H(\cdot)$).

Similarly to [12] we can argue that the view of A and thus
 its outputs have the same distribution in Games 3 and 4 unless
 A or the decryption oracle queries the H oracle on either of
 the points $s_1^+, \dots, s_{q_e}^+$. Let's denote the probability of such
 event in this game $\Pr[\text{AskH}_4]$, and such an event is defined
 similarly in the following games.

$$\Pr[\text{AskG}_3] - \Pr[\text{AskG}_4] \leq \Pr[\text{AskH}_4]$$

and

$$\Pr[\text{AskG}_4] \leq \frac{q_e(q_G + q_d)}{2^{k_0}}.$$

Game 5 is similar to Game 4 except the way the challenge
 ciphertexts are generated. In this game they are simply picked
 at random, independently from everything else. We can argue
 similarly to the proof in [12] that this does not change the view
 and the outputs of A . The reason is that f_k is a permutation
 and in Game 4 it was applied to uniformly distributed points
 $s_i^* \parallel t_i^*$, where $s_i^* = s_i^+$ and $t_i^* = h_i^+ \oplus r_i^+$.

$$\Pr[\text{AskH}_4] = \Pr[\text{AskH}_5].$$

Note that the pre-computed values $r_i^+, g_i^+, s_i^+, h_i^+$ for $1 \leq$
 $i \leq q_e$ are not used further in the game any more. Therefore
 in the following games we remove the code generating them
 for convenience.

Games 6–8 deal with answering decryption oracle queries
 which were simulated perfectly before that. The definitions of
 the games and their analysis done in [12] hold for our modified
 scheme and are independent of the number of the challenge
 encryption oracle queries A does, but we describe them for
 completeness. For a k_2 -bit ciphertext C we call its last k_0 bits
 t , and the first $k_2 - k_0$ bits of $f_k^{-1}(C)$ we call s . We call r the
 result of xoring t with $H(s)$.

Game 6 is like Game 5 except the decryption oracle rejects
 all ciphertexts for which the underlying r -value has not been
 previously queried to the G oracle by the adversary. The views
 of A in Games 5 and 6 are different only if A queries a valid
 ciphertext without querying the underlying r -value to G oracle.
 A ciphertext is valid if the last k_1 bits of $s \oplus G(r)$ are zeros.
 But if r has not been queried, then $G(r)$ is an independent
 random string and validity will be satisfied with probability at
 most 2^{-k_1} . For q_d decryption queries we get

$$\Pr[\text{AskH}_5] - \Pr[\text{AskH}_6] \leq \frac{q_d}{2^{k_1}}.$$

Game 7 is like Game 6 except that the decryption oracle
 rejects all ciphertexts for which the underlying s -value has not
 been previously queried to the H oracle by the adversary. The
 views of A in Games 6 and 7 are different only if A queries
 a valid ciphertext without querying the underlying s -value to
 H oracle when the query r was made to the G oracle. Since

$r = H(s) \oplus t$, $H(s)$ was not previously defined, it is random and independent. Hence the probability that r was queried is at most $q_G/2^{k_0}$. And for q_d decryption queries we get

$$\Pr[\text{AskH}_6] - \Pr[\text{AskH}_7] \leq \frac{q_d q_G}{2^{k_0}}.$$

In the last Game 8 the decryption oracle queries, for which either of the corresponding r and s values have not been queried, are rejected. The game stores the pairs of the random oracle queries and the corresponding answers in the arrays G-list and H-list. The other ciphertexts are decrypted by using a simple plaintext extractor who expects all previously made G and H queries made by A and returns the matching plaintext. Namely, to decrypt a ciphertext $C \parallel t$, take every $(s_j, H(s_j))$ in H-list for $1 \leq j \leq q_H$, compute $r \leftarrow H(s_j) \oplus t$ and check if there is a pair (r, G_r) for any G_r in the G-list. If so, check if $f_k(s_j \parallel t) = C \parallel t$ and the last k_1 bits of $s_j \oplus G_r$ are zeros, then return the rest of $s_j \oplus G_r$. Otherwise, return \perp .

The view of the adversary does not change and thus

$$\Pr[\text{AskH}_7] = \Pr[\text{AskH}_8].$$

Putting it all together we get

$$\begin{aligned} \frac{1}{2} \cdot \mathbf{Adv}_{\text{OAEP}[\mathcal{F}_k], A}^{\text{ind-cca-mq}}(k) &= \Pr[\text{Game}_0] - \frac{1}{2} \\ &= \Pr[\text{Game}_1] - \frac{1}{2} \\ &\leq \Pr[\text{Game}_2] - \frac{1}{2} + \Pr[\text{AskG}_2] \\ &\leq \Pr[\text{AskG}_2] \\ &= \Pr[\text{AskG}_3] \\ &\leq \Pr[\text{AskG}_4] + \Pr[\text{AskH}_4] \\ &\leq \frac{q_e(q_G + q_d)}{2^{k_0}} + \Pr[\text{AskH}_5] \\ &\leq \frac{q_e(q_G + q_d)}{2^{k_0}} + \frac{q_d}{2^{k_1}} + \Pr[\text{AskH}_6] \\ &\leq \frac{q_e(q_G + q_d) + q_d q_G}{2^{k_0}} \\ &+ \frac{q_d}{2^{k_1}} + \Pr[\text{AskH}_7] \\ &= \frac{q_e(q_G + q_d) + q_d q_G}{2^{k_0}} \\ &+ \frac{q_d}{2^{k_1}} + \Pr[\text{AskH}_8]. \end{aligned}$$

PROOF CONCLUSION FOR THEOREM 4.1. We show that there exists an adversary B such that

$$\Pr[\text{AskH}_8] \leq q_e \cdot \mathbf{Adv}_{\mathcal{F}_k, B}^{\text{owf}}(k). \quad (2)$$

B is given f and the challenge y . It first picks a random index j from $\{1, \dots, q_e\}$. Next B picks q_e values at random from $\{0, 1\}^k$, let us call them v_1, \dots, v_{q_e} ; and q_e values at random from $\{0, 1\}^{p-k}$, let us call them w_1, \dots, w_{q_e} . B re-assigns y to v_j . B runs A on input public key f answers its RO queries with random and independent values (and records all queries and

answers). To answer the decryption oracle queries B checks if the corresponding G and H queries were made, and in this case a simple plaintext extractor that uses the recorded queries to the random oracles and the (random) answers as we described above is used; otherwise, the ciphertexts are rejected (B returns \perp). For $1 \leq i \leq q_e$ for an i -th query to the challenge oracle made by A , B returns $v_i \parallel w_i$.

We claim that B simulates the view of A in Game 8 perfectly. The challenge ciphertexts are random and independent strings, and the decryption queries are answered correctly by the plaintext extractor. Event AskH_8 means that A made a query h to the random oracle H so that $f(h[1, \dots, k]) = v_i$ for some $1 \leq i \leq q_e$. B searches for such query and outputs $h[1, \dots, k]$, which is $f^{-1}(y)$, if $i = j$. Therefore, B breaks one-wayness of F with probability $1/q_e$, since the index j was picked at random from $\{1, \dots, q_e\}$ independently from A 's view.

We finally justify the running time of B . In addition to running A , in order to answer each of q_d decryption queries B does one f computation for each of q_H pairs stored in H-list. The reason why the bound in the proof in [12] for the original OAEP[\mathcal{F}] is worse (regardless of the fact that the reduction there is to an easier problem of partial-domain one-wayness) is because there for each decryption oracle query the adversary has to apply the random instance f to values corresponding to all possible combinations of pairs stored in G-list and H-list or use more storage to speed up the check. But in any case it stays proportional to $q_G \cdot q_H$. In our case we do not need to test every r -value in G-list as the required value can be computed using the t -part of the ciphertext.

PROOF CONCLUSION FOR THEOREM 5.1. We now consider RSA trapdoor permutation generator in place of F and claim that there exists an adversary B such that

$$\Pr[\text{AskH}_8] \leq \mathbf{Adv}_{\text{RSA}, B}^{\text{owf}}(k). \quad (3)$$

This is where we use self-reducibility of RSA to improve tightness of the reduction. (Random) self-reducibility informally means that given a successful algorithm that can solve a problem for an average case one can also solve the problem in the worst case. We use this property to incorporate a single challenge for the simulator into multiple challenges for the adversary the simulator runs. Regardless of which challenge the adversary breaks, the simulator can break its own challenge.

To justify Equation (3) we construct B as follows. B is given an RSA public key (N, e) and a challenge $y = x^e \pmod N$ for a random $x \in \mathbb{Z}_N^*$. B picks q_e values at random from \mathbb{Z}_N^* , let us call them v_1, \dots, v_{q_e} ; and q_e values at random from $\{0, 1\}^{p-k}$, let us call them w_1, \dots, w_{q_e} . B runs A on public key (N, e) , answers its RO queries with random and independent values (and records all queries and answers). To answer the decryption oracle queries B checks if the corresponding G and H queries were made, and in this case a simple plaintext extractor we described above is used; otherwise, the ciphertexts are rejected (B returns \perp). For $1 \leq i \leq q_e$ for an i -th query to the challenge oracle made by A , B returns $(yv_i^e \pmod N) \parallel w_i$.

We claim that B simulates the view of A in Game 8 perfectly (except for the mismatch between the sets \mathbb{Z}_N^* and $\{0, 1\}^k$, on which we comment in Section VII). The challenge ciphertexts are random and independent strings, and the decryption queries are answered according to the simple plaintext extractor algorithm that uses the recorded queries to the random oracles and the (random) answers. Event AskH₈ means that A made a query h to the random oracle G so that $h[1, \dots, k]^e = yv_j^e \pmod{N}$ for some $1 \leq j \leq q_e$. B searches for such query and outputs $h[1, \dots, k]v_j^{-1} \pmod{N}$, which is $y^d \pmod{N}$, i.e. it breaks one-wayness of RSA.

We finally justify the running time of B . In addition to running A , to compute each of q_e challenge ciphertexts B does one modulo multiplication (note that under our convention we do not have to count RSA computations in encryption as they are part of the A overlying experiment), to answer each of q_d decryption queries B does one RSA computation for each of q_H pairs stored in H-list, and does one modulo inverse and one multiplication at the end. ■

REMARK. We comment on why does not the above proof showing the security improvement work for the unmodified OAEP[RSA] scheme. The reason is that in the original scheme the RSA permutation is applied to the whole output $s \parallel t$ of the OAEP transform. The tight security of OAEP[RSA] is only shown assuming partial-domain one-wayness of RSA. In the proof above the adversary B given a challenge y could still use self-reducibility of RSA and generate challenge ciphertexts for A as $yv_1^e, \dots, yv_{q_e}^e \pmod{N}$. In A 's view, these ciphertexts have the right distribution (in Game 8) unless A queries the H oracle on any of the underlying s values (the first part of $y^d v_1, \dots, y^d v_{q_e}$). But if this happens, B cannot compute y^d , as it does not know the remaining part of the transform.

VI. ENCRYPTING LONG MESSAGES WITH MODIFIED RSA-OAEP

We observe that the modified RSA-OAEP scheme can be used to encrypt long messages without employing symmetric encryption in the hybrid encryption construct. For that the function G in the transform is made variable-output-length, i.e. its output is the length of the message plus the zero padding of length k_1 . For a fixed-output-length hash $G'(\cdot)$ one can efficiently construct $G(\cdot)$ as $G'(\langle 0 \rangle \parallel \cdot) \parallel G'(\langle 1 \rangle \parallel \cdot) \dots \parallel G'(\langle l \rangle \parallel \cdot)$, where $\langle i \rangle$ means the binary representation of the counter $i \in \mathbb{N}$. In the RO model G is a random oracle if G' is. The function H in the transform needs to be variable-input-length, which is not a problem, since most of the hash functions are. The RSA function is applied to the first k (e.g. 1024) bits of the s -part of the OAEP transform. The proof of security stays virtually the same. This scheme yields more compact ciphertexts for long messages than the one obtained through the use of hybrid encryption because there is no need to encrypt the symmetric key.

VII. ON $\{0, 1\}^k$ VS \mathbb{Z}_N^* MISMATCH IN THE PROOF OF THEOREM 5.1

In the proof of Theorem 5.1 we mostly ignored the fact that the first part of the OAEP output, to which RSA function

is applied, is distributed uniformly in $\{0, 1\}^k$, while the pre-image of the RSA challenge in the definition of one-wayness (cf. Definition 2.2) is a random element of \mathbb{Z}_N^* , where $N < 2^k$ is part of the public key. Keeping this discrepancy implicit is common to the security proofs for RSA-OAEP and many other RSA-based schemes. One solution for RSA-OAEP was pointed out in [12]. Namely, a sender just re-computes the output of the OAEP transform with new random coins until it falls in \mathbb{Z}_N^* . The expected number of transform computations is very low, and thus this solution is practical. However, one has to remember that the running time of the encryption algorithm and the adversary in the reduction is now *expected* polynomial time.

We show how can this discrepancy be eliminated for RSA-OAEP++ without relying on the success of multiple re-computations, by slightly tweaking the transform as follows. We introduce an additional parameter to the scheme, $k_3 \in \mathbb{N}$ so that $k_3 \gg \log_2 q_e$, and make the random oracle G be $\{0, 1\}^{k_0} \rightarrow \{0, 1\}^{k_2 - k_0 + k_3}$, i.e. we extend the length of the output by k_3 bits. As before, $H: \{0, 1\}^{k_2 - k_0} \rightarrow \{0, 1\}^{k_0}$ and the message space is $\{0, 1\}^{k_2 - k_0 - k_1}$. The encryption algorithm parses $M \parallel 0^{k_1}$ as $w \parallel y$, where $|w| = k$ and $|y| = k_2 - k_0 - k$, picks $r \xleftarrow{\$} \{0, 1\}^{k_0}$ and parses the output of $G(r)$ as $u \parallel v$, where $|u| = k + k_3$ and $|v| = k_2 - k_0 - k$. It finally computes $z \leftarrow w + u \pmod{N}$ and $s \leftarrow y \oplus v$, $t \leftarrow H(z \parallel s) \oplus r$ and returns $f_k(z \parallel s \parallel t)$. Here N is part of the public key description RSA_k . The decryption algorithm is defined accordingly.

The proof of Theorem 5.1 holds but we need to show that u , and thus z , is distributed almost uniformly in \mathbb{Z}_N (and hence for any efficient adversary this distribution is indistinguishable from a uniform distribution in \mathbb{Z}_N^*). Let $2^{k+k_3} = aN + b$, where $a = \lfloor (2^{k+k_3}/N) \rfloor$ and $b = 2^{k+k_3} \pmod{N}$. Note that since $N < 2^k$ we have that $a \geq 2^{k_3}$.

Let $\$$ denote a random variable uniformly distributed in \mathbb{Z}_N . Then the statistical distance between u and $\$$ is

$$\begin{aligned} \Delta(u, \$) &= b \left(\frac{a+1}{aN+b} - \frac{1}{N} \right) \\ &+ (N-b) \left(\frac{1}{N} - \frac{a}{aN+b} \right) \\ &= \frac{2bN - 2b^2}{(aN+b)N} \tag{4} \\ &= \frac{2bN - 2b^2}{2^{k+k_3}N}. \tag{5} \end{aligned}$$

It is easy to see that Equation 5 achieves the maximum when $b = N/2$. Plugging it into Equation 4 we get

$$\Delta(u, \$) = \frac{1}{2a+1} \leq \frac{1}{2^{k_3+1} + 1}.$$

Since we have q_e ciphertexts,

$$q_e \Delta(u, \$) \leq \frac{q_e}{2^{k_3+1} + 1}$$

should be negligible, so k_3 should be chosen large enough compared to $\log_2 q_e$.

VIII. CONCLUSIONS

We suggested a slight modification of the well-known and practical RSA-OAEP encryption scheme. We showed that this scheme has extra advantages, namely its IND-CCA security remains tightly related (in the RO model) to hardness of the RSA problem, even in the multi-query setting. Additionally, this scheme can be used for encryption of long messages without employing the hybrid encryption method and symmetric encryption. We believe the modification is very simple to implement and may be considered by the standard bodies.

IX. ACKNOWLEDGMENTS

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