Handbook of Robotics

Chapter 1: Kinematics

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September 15, 2005

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Chapter 1

Kinematics

Kinematics is the geometry of motion, which is to say geometry with the addition of the dimension of time. It pertains to the position, velocity, acceleration, and all higher order derivatives of the position of a body in space without regard to the forces that cause the motion of the body. Since robotic mechanisms are by their very essence designed for motion, kinematics is the most fundamental aspect of robot design, analysis, control, and simulation.

Unless explicitly stated otherwise, the kinematic description of robotic mechanisms typically employs a number of idealizations. The links that compose the robotic mechanism are assumed to be perfectly rigid bodies having surfaces that are geometrically perfect in both position and shape. Accordingly, these rigid bodies are connected together at joints where their idealized surfaces are in complete contact without any clearance between them. The respective geometries of these surfaces in contact determine the freedom of motion between the two links, or the **joint kinematics**. In an actual robotic mechanism, these joints will have some physical limits beyond which motion is prohibited. The **workspace** of a robotic manipulator is determined by considering the combined limits and freedom of motion of all of the joints within the mechanism.

A common task for a robotic manipulator is to locate its end-effector in a specific position and with a specific orientation within its workspace. Therefore, it is critical to have a representation of the position and orientation of a body, such as an end-effector, in space. While there are in principle an infinite number of ways to describe the position and orientation of a body in space, this chapter will provide an overview of the representations that are particularly convenient for the kinematic analysis of robotic mechanisms. Regardless of the selected representation, two fundamental kinematic problems arise for robotic manipulators. **Forward kinematics** is the determination of the position and orientation of the end-effector from the specified values of the manipulator's joint variables. **Inverse kinematics** is the determination of the values of the manipulator's joint variables required to locate the end-effector in space with a specified position and orientation. The problems of **forward** and **inverse instantaneous kinematics** are analogous, but address velocities rather than positions. While kinematics does not consider the forces that generate motion, kinematic velocity analysis is closely related to static wrench analysis. The **Jacobian** that maps the joint velocities to the end-effector velocity also maps the end-effector's **transmitted static wrench** to the joint forces/torques that produce the wrench.

The goal of this chapter is to provide the reader with an overview of different approaches to representing the position and orientation of a body in space and algorithms to use these approaches in solving the problems of forward kinematics, inverse kinematics, and static wrench transmission for robotic manipulators.

1.1 Introduction

It is necessary to treat problems of the description and displacement of systems of rigid bodies since robotic mechanisms are systems of rigid bodies connected by kinematic joints. Among the many possible topologies in which systems of bodies can be connected, two are of particular importance in robotics: serial chains and fully parallel mechanisms. A serial chain is a system of rigid bodies in which each member is connected to two others, except for the first and last members that are each connected to only one other member. A fully parallel mechanism is one in which there are two members that are connected together by multiple joints. In practice, each "joint" is often itself a serial chain. This chapter focuses almost exclusively on serial chains. Parallel mechanisms are dealt with in more detail in Chapter 13 Kinematic

Structure and Analysis.

Obviously, the draft of this chapter in its current form is incomplete. It is missing figures and tables. It is short on references in many places. The later sections are particularly inadequate in their present form. This Introduction section itself is essentially yet to be written.

1.2 Position and Orientation Representation

Spatial, rigid body kinematics can be viewed as a comparative study of different ways of representing the position and orientation of a body in space. Translations and rotations, referred to in combination as rigid body displacements, are also expressed with these representations. No one approach is optimal for all purposes, but the advantages of each can be leveraged appropriately to facilitate the solution of different problems.

The minimum number of coordinates required to locate a body in Euclidean space is six: three for position and three for orientation. Many representations of spatial position and orientation employ sets with superabundant coordinates in which auxiliary relationships exist among the coordinates. The number of independent auxiliary relationships is the difference between the number of coordinates in the set and six.

This chapter and those that follow it make frequent use of "coordinate reference frames" or simply "frames". A coordinate reference frame i consists of an origin, denoted O_i, and a triad of mutually orthogonal basis vectors, denoted $[\hat{\boldsymbol{x}}_i \; \hat{\boldsymbol{y}}_i \; \hat{\boldsymbol{z}}_i]$, that are all fixed within a particular body. The position and orientation of a body in space will always be expressed relative to some other body, so they can be expressed as the position and orientation of one coordinate frame relative to another. Similarly, rigid body displacements can be expressed as displacements between two coordinate frames, one of which may be referred to as "moving", while the other may be referred to as "fixed". This simply indicates that the observer is located in a stationary position within the fixed reference frame, not that there exists any absolutely fixed frame.

1.2.1 Representation of Position

The position of the origin of coordinate frame i relative to coordinate frame j can be denoted by the 3×1 vector ${}^j\boldsymbol{p}_i = \left[{}^jp_i^x \; {}^jp_i^y \; {}^jp_i^z\right]^T$. The components of this vector are the Cartesian coordinates of O_i in the j frame, which are the projections of the vector ${}^j\boldsymbol{p}_i$ onto the corresponding axes. The vector components could also be expressed as the spherical or cylindrical coordinates of O_i in the j frame. Such representations have advantages for analysis of robotic mechanisms including spherical and cylindrical joints.

A translation is a displacement in which no point in the rigid body remains in its initial position and all straight lines in the rigid body remain parallel to their initial orientations. The translation of a body in space can be represented by the combination of its position prior to the translation and its position following the translation. Conversely, the position of a body can be represented as a translation that takes the body from a position in which the coordinate frame fixed to the body coincides with the fixed coordinate frame to the current position in which the two fames are not coincident. Thus, any representation of position can be used to create a representation of displacement, and vice-versa.

1.2.2 Representation of Orientation

There is significantly greater breadth in the representation of orientation than in that of position. This section does not include an exhaustive summary, but focuses on the representations most commonly applied to robotic manipulators.

A rotation is a displacement in which at least one point of the rigid body remains in its initial position and not all lines in the body remain parallel to their initial orientations. The points and lines addressed here and in the definition of translation above are not necessarily contained within the boundaries of the finite rigid body. Rather, any point or line in space can be taken to be rigidly fixed in a body. For example, a body in a circular orbit rotates about an axis through the center of its circular path, and every point on the axis of rotation is a point in the body that remains in its initial position despite not being physically within the boundaries of the body. As in the case of position and translation, any representation of orientation can be used to create a representation of rotation, and vice-versa.

Rotation Matrices

The orientation of coordinate frame i relative to coordinate frame j can be denoted by expressing the basis vectors $[\hat{\boldsymbol{x}}_i \; \hat{\boldsymbol{y}}_i \; \hat{\boldsymbol{z}}_i]$ in terms of the basis vectors $\left[\hat{\boldsymbol{x}}_j \; \hat{\boldsymbol{y}}_j \; \hat{\boldsymbol{z}}_j\right]$. This yields $\left[{}^j\hat{\boldsymbol{x}}_i \; {}^j\hat{\boldsymbol{y}}_i \; {}^j\hat{\boldsymbol{z}}_i\right]$, which when written together as a 3×3 matrix is known as the rotation matrix. The components of ${}^j\boldsymbol{R}_i$ are the dot products of basis vectors of the two coordinate frames.

$$ {}^j\boldsymbol{R}_i = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j & \hat{\boldsymbol{y}}_i \cdot \hat{\boldsymbol{x}}_j & \hat{\boldsymbol{z}}_i \cdot \hat{\boldsymbol{x}}_j \\ \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{y}}_j & \hat{\boldsymbol{y}}_i \cdot \hat{\boldsymbol{y}}_j & \hat{\boldsymbol{z}}_i \cdot \hat{\boldsymbol{y}}_j \\ \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{z}}_j & \hat{\boldsymbol{y}}_i \cdot \hat{\boldsymbol{z}}_j & \hat{\boldsymbol{z}}_i \cdot \hat{\boldsymbol{z}}_j \end{bmatrix} \tag{1.1} $$

Because the basis vectors are unit vectors and the dot product of any two unit vectors is the cosine of the angle between them, the components are commonly referred to as direction cosines.

The rotation matrix ${}^j\boldsymbol{R}_i$ contains nine elements, while only three parameters are required to define the orientation of a body in space. Therefore, six auxiliary relationships exist between the elements of the matrix. Because the basis vectors of coordinate frame i are mutually orthonormal, as are the basis vectors of coordinate frame j, the columns of ${}^j\boldsymbol{R}_i$ formed from the dot products of these vectors are also mutually orthonormal. A matrix composed of mutually orthonormal vectors is known as an orthogonal matrix and has the property that its inverse is simply its transpose. This property provides the six auxiliary relationships. Three require the column vectors to have unit length, and three require the column vectors to be mutually orthogonal. Alternatively, the orthogonality of the rotation matrix can be seen by considering the frames in reverse order. The orientation of coordinate frame j relative to coordinate frame i is the rotation matrix ${}^i\boldsymbol{R}_j$ whose rows are clearly the columns of the matrix ${}^j\boldsymbol{R}_i$. Rotation matrices are combined through simple matrix multiplication such that the orientation of frame i relative to frame k can be expressed as ${}^k\boldsymbol{R}_i = {}^k\boldsymbol{R}_j {}^j\boldsymbol{R}_i$.

In summary, ${}^j\boldsymbol{R}_i$ is the rotation matrix that transforms a vector expressed in coordinate frame i to a vector expressed in coordinate frame j. It provides a representation of the orientation of frame i relative to j and thus, can be a representation of rotation from frame i to frame j. Table 1.1 lists the equivalent rotation matrices for the other representations of orientation listed in this section. Table 1.2 contains the conversions from a known rotation matrix to these other representations.

Euler Angles

For a minimal representation, the orientation of coordinate frame i relative to coordinate frame j can be denoted as a vector of three angles $[\alpha, \beta, \gamma]$. These angles are known as Euler angles when each represents a rotation about an axis of a moving coordinate frame. In this way, the location of the axis of each successive rotation depends upon the preceding rotation(s), so the order of

Z-Y-X Euler Angles $[\alpha, \beta, \gamma]$:

$$\begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

X-Y-Z Fixed Angles $[\psi, \theta, \phi]$:

$$\begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Angle-Axis $\theta\hat{\mathbf{K}}$ (where $v_\theta = 1 - c_\theta$):

$$\begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

Unit Quaternions $[q_0, q_1, q_2, q_3]$:

$$\begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix}$$

Table 1.1: Equivalent rotation matrices for various representations of orientation.

the rotations must accompany the three angles to define the orientation. For example, the symbols $[\alpha, \beta, \gamma]$ are used throughout this handbook to indicate Z-Y-X Euler angles. α is the rotation about the $\hat{\boldsymbol{z}}$ axis, β is the rotation about the rotated $\hat{\boldsymbol{y}}$ axis, and finally, γ is the rotation about the twice rotated $\hat{\boldsymbol{x}}$ axis. Z-Y-Z Euler angles are another commonly used convention from among the 12 different possible orders of rotations.

Fixed Angles

A vector of three angles can also denote the orientation of coordinate frame i relative to coordinate frame j when each angle represents a rotation about an axis of a fixed reference frame. Appropriately, such angles are referred to as Fixed Angles, and the order of the rotations must again accompany the angles to define the orientation. X-Y-Z Fixed Angles, denoted here as $[\psi, \theta, \phi]$, are a common convention from among the, again, 12 different possible orders of rotations. ψ is the yaw rotation about the fixed $\hat{\boldsymbol{x}}$ axis, θ is the pitch rotation about the fixed $\hat{\boldsymbol{y}}$ axis, and ϕ is the roll rotation about the fixed $\hat{\boldsymbol{z}}$ axis. As can

Z-Y-X Euler Angles $[\alpha, \beta, \gamma]$:

$$\beta = Atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$\alpha = Atan2\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)$$
$$\gamma = Atan2\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)$$

X-Y-Z Fixed Angles $[\psi, \theta, \phi]$:

$$\theta = Atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$\psi = Atan2\left(\frac{r_{21}}{c_\theta}, \frac{r_{11}}{c_\theta}\right)$$
$$\phi = Atan2\left(\frac{r_{32}}{c_\theta}, \frac{r_{33}}{c_\theta}\right)$$

Angle-Axis $\theta\hat{\mathbf{K}}$:

$$\theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$
$$\hat{\mathbf{K}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{21} \end{bmatrix}$$

Unit Quaternions $[q_0, q_1, q_2, q_3]$:

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$
$$q_1 = \frac{r_{32}-r_{23}}{4q_0}$$
$$q_2 = \frac{r_{13}-r_{31}}{4q_0}$$
$$q_3 = \frac{r_{21}-r_{12}}{4q_0}$$

Table 1.2: Conversions from a rotation matrix to various representations of orientation.

be seen by comparing the respective equivalent rotation matrices in Table 1.1 and the respective conversions in Table 1.2, a set of X-Y-Z Fixed Angles is exactly equivalent to the same set of Z-Y-X Euler Angles ($\alpha = \phi$, $\beta = \theta$, and $\gamma = \psi$). This result holds in general such that three rotations about the three axes of a fixed frame define the same orientation as the same three rotations taken in the opposite order about the three axes of a moving frame.

Angle-Axis

A single angle θ in combination with a unit vector $\hat{\mathbf{K}}$ can also denote the orientation of coordinate frame i relative to coordinate frame j. In this case, frame i is rotated through the angle θ about an axis defined by the vector $\hat{\mathbf{K}} = [k_x\ k_y\ k_z]^T$ relative to frame j. The vector $\hat{\mathbf{K}}$ is sometimes referred to as the equivalent axis of a finite rotation. The angle-axis representation, typically written as either $\theta\hat{\mathbf{K}}$ or $[\theta k_x\ \theta k_y\ \theta k_z]^T$, is superabundant by one

Angle-Axis to Unit Quaternions:

$$q_0 = \cos\frac{\theta}{2}$$
$$q_1 = k_x \sin\frac{\theta}{2}$$
$$q_2 = k_y \sin\frac{\theta}{2}$$
$$q_3 = k_z \sin\frac{\theta}{2}$$

Quaternions to Angle-Axis:

$$\theta = 2\cos^{-1} q_0$$
$$k_x = \frac{q_1}{\sin\frac{\theta}{2}}$$
$$k_y = \frac{q_2}{\sin\frac{\theta}{2}}$$
$$k_z = \frac{q_3}{\sin\frac{\theta}{2}}$$

Table 1.3: Conversions from angle-axis to unit quaternion representations of orientation and vice versa.

because it contains four parameters. The auxiliary relationship that resolves this is the unit magnitude of vector $\hat{\mathbf{K}}$. Even with this auxiliary relationship, the angle-axis representation is not unique because rotation through an angle of $-\theta$ about $-\hat{\mathbf{K}}$ is equivalent to a rotation through θ about $\hat{\mathbf{K}}$. Table 1.3 contains the conversions from angle-axis representation to unit quaternions and vice versa. The conversions from these two representations to Euler angles or fixed angles can be easily found by using the conversions in Table 1.2 in conjunction with the equivalent rotation matrices in Table 1.1.

Quaternions

Quaternions provide a representation of the orientation of one coordinate frame relative to another that in some ways corresponds to complex numbers. A quaternion $\mathbf{q}$ is defined to have the form, $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$, where q_0, q_1, q_2, and q_3 are scalars, sometimes referred to as Euler parameters, and i, j, and k are operators somewhat analogous to the operator $i = \sqrt{-1}$ used in complex numbers. The operators are defined to satisfy the following combinatory rules: $ii = jj = kk = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$. Two quaternions are added by adding the respective scalar elements separately, so the operators act as separators. The null element for addition is the quaternion $\mathbf{0} = 0 + 0i + 0j + 0k$, and quaternion sums are associative, commutative, and distributive. The null element for multiplication is $\mathbf{I} = 1 + 0i + 0j + 0k$, as can be seen using $\mathbf{Iq} = \mathbf{q}$ for any quaternion $\mathbf{q}$. Quaternion products are associative and distributive, but not commutative, and following the conventions of the operators and addition, have the form,

$$\mathbf{ab} = \begin{array}{l} a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ +(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i \\ +(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)j \\ +(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)k \end{array}. \tag{1.2}$$

It is convenient to define the conjugate of a quaternion, $\tilde{\mathbf{q}} = q_o - q_1 i - q_2 j - q_3 k$, so that $\mathbf{q}\tilde{\mathbf{q}} = \tilde{\mathbf{q}}\mathbf{q} = q_0^2 + q_1^2 + q_3^2$. A unit quaternion can then be defined such that, $\mathbf{q}\tilde{\mathbf{q}} = 1$. Often, q_0 is referred to as the scalar part of the quaternion, and $[q_1\ q_2\ q_3]^T$ is referred to as the vector part.

Since a quaternion contains four elements, it is superabundant by one for describing orientation, and the unit magnitude of a unit quaternion provides the auxiliary relationship to resolve this. A vector is defined in quaternion notation as a quaternion with zero real part. Thus, a vector $\boldsymbol{p} = [p_x\ p_y\ p_z]^T$ can be expressed as a quaternion $\mathbf{p} = p_x i + p_y j + p_z k$. For any unit quaternion $\mathbf{q}$, the operation $\mathbf{q}\mathbf{p}\tilde{\mathbf{q}}$ performs a rotation of the vector $\boldsymbol{p}$ about the direction $[q_1\ q_2\ q_3]^T$. This is clearly seen by expanding the operation $\mathbf{q}\mathbf{p}\tilde{\mathbf{q}}$ and comparing the results with the equivalent rotation matrix listed in Table 1.1.

1.2.3 Homogeneous Transformations

The preceding sections have addressed representations of position and orientation separately. With homogeneous transformations, position vectors and rotation matrices are combined together in a compact notation. Any vector ${}^i\boldsymbol{\rho}$ expressed relative to the i coordinate frame can be expressed relative to the j coordinate frame if the position and orientation of the i frame are known relative to the j frame. Using the notation of Section 1.2.1, the position of the origin of coordinate frame i relative to coordinate frame j can be denoted by the vector ${}^j\boldsymbol{p}_i = \left[{}^jp_i^x\ {}^jp_i^y\ {}^jp_i^z\right]^T$. Using the notation of Section 1.2.2, the orientation of frame i relative to frame j can be denoted by the rotation matrix ${}^j\boldsymbol{R}_i$. Thus,

$${}^j\boldsymbol{\rho} = {}^j\boldsymbol{R}_i{}^i\boldsymbol{\rho} + {}^j\boldsymbol{p}_i. \tag{1.3}$$

This equation can be written,

$$\begin{bmatrix} {}^j\boldsymbol{\rho} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^j\boldsymbol{R}_i & {}^j\boldsymbol{p}_i \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right] \begin{bmatrix} {}^i\boldsymbol{\rho} \\ 1 \end{bmatrix}, \tag{1.4}$$

where,

$${}^j\boldsymbol{T}_i = \left[\begin{array}{c|c} {}^j\boldsymbol{R}_i & {}^j\boldsymbol{p}_i \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right], \tag{1.5}$$

is the 4×4 homogeneous transform matrix and $\left[{}^j\boldsymbol{\rho}\ 1\right]^T$ and $\left[{}^i\boldsymbol{\rho}\ 1\right]^T$ are the homogeneous representations of the position vectors ${}^j\boldsymbol{\rho}$ and ${}^i\boldsymbol{\rho}$. The matrix ${}^j\boldsymbol{T}_i$ transforms vectors from coordinate frame i to coordinate frame j. Its inverse, ${}^j\boldsymbol{T}_i^{-1}$ transforms vectors from coordinate frame j to coordinate frame i.

$${}^j\boldsymbol{T}_i^{-1} = {}^i\boldsymbol{T}_j = \left[\begin{array}{c|c} {}^j\boldsymbol{R}_i^T & -{}^j\boldsymbol{R}_i^T\,{}^j\boldsymbol{p}_i \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]. \tag{1.6}$$

Composition of 4×4 homogeneous transform matrices is accomplished through simple matrix multiplication, just as in the case of 3×3 rotation matrices. Therefore, ${}^k\boldsymbol{T}_i = {}^k\boldsymbol{T}_j{}^j\boldsymbol{T}_i$. Since matrix multiplications do not commute, the order or sequence is important.

Homogeneous transformations are particularly attractive when compact notation is desired and/or when ease of programming is the most important consideration. This is not, however, a computationally efficient representation since it introduces a large number of additional multiplications by ones and zeros. Although homogeneous transform matrices technically contain sixteen elements, four are defined to be zero or one, and the remaining elements are composed of a rotation matrix and a position vector. Therefore, the only truly superabundant coordinates come from the rotation matrix component, so the relevant auxiliary relationships are those associated with the rotation matrix.

1.2.4 Screw Transformations

The transformation in Equation 1.3 can be viewed as composed of a rotation between coordinate frames i and j and a separate displacement between those frames. To get from frame i to frame j, one could perform the rotation first, followed by the displacement, or vice versa. Alternatively, the spatial displacement between the frames can be expressed, except in the case of a pure translation, as a rotation about a unique line combined with a translation parrallel to that line. Since this is precisely the type of displacement produced by a screw, the line about which rotation takes place is referred to as the screw axis. The displacement itself is often called the twist about that screw axis, and the ratio of the linear displacement d to the rotation θ is referred to as the pitch h of the screw axis. Thus, $d = h\theta$. In a screw displacement, every point in the body has the same linear displacement d parallel to the screw axis. Each point is also rotated about that axis through the same angle θ. Points that lie on the screw axis remain on the axis and are displaced a distance d along it. The screw axis of a

Screw Transformation to Homogeneous Transformation (where $v_\theta = 1 - c_\theta$):

$$
{}^j\boldsymbol{R}_i = \begin{bmatrix} w_x^2 v_\theta + c_\theta & w_x w_y v_\theta - w_z s_\theta & w_x w_z v_\theta + w_y s_\theta \\ w_x w_y v_\theta + w_z s_\theta & w_y^2 v_\theta + c_\theta & w_y w_z v_\theta - w_x s_\theta \\ w_x w_z v_\theta - w_y s_\theta & w_y w_z v_\theta + w_x s_\theta & w_z^2 v_\theta + c_\theta \end{bmatrix}
$$

$$
{}^j\boldsymbol{p}_i = \left(\mathbf{I} - {}^j\boldsymbol{R}_i\right)\boldsymbol{\rho} + h\theta\hat{\boldsymbol{w}}
$$

Homogeneous Transformation to Screw Transformation:

$$
\boldsymbol{l} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}^T
$$

$$
\theta = sign\left(\boldsymbol{l}^T\, {}^j\boldsymbol{p}_i\right)\left|\cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)\right|
$$

$$
h = \frac{\boldsymbol{l}^T\, {}^j\boldsymbol{p}_i}{2\theta\sin\theta}
$$

$$
\boldsymbol{\rho} = \frac{\left(\mathbf{I} - {}^j\boldsymbol{R}_i^T\right){}^j\boldsymbol{p}_i}{2(1-\cos\theta)}
$$

$$
\hat{\boldsymbol{w}} = \frac{\boldsymbol{l}}{2\sin\theta}
$$

Table 1.4: Conversions from a screw transformation to a homogeneous transformation and vice versa.

pure translation is not unique. Any line parallel to the translation can be regarded as the screw axis, and since the rotation θ is zero, the axis of a translation is said to have infinite pitch.

A screw axis is most conveniently represented in any reference frame by means of a unit vector $\hat{\boldsymbol{w}}$ parallel to it and the position vector $\boldsymbol{\rho}$ of any point lying on it. Additional specification of the pitch h and the angle of rotation θ completely defines the location of a second coordinate frame relative to the reference frame. Thus, a total of eight coordinates define a screw transformation, which is superabundant by two. The unit magnitude of vector $\hat{\boldsymbol{w}}$ provides one auxiliary relationship, but in general, there is no second auxiliary relationship because the same screw axis is defined by all points lying on it, which is to say that the vector $\boldsymbol{\rho}$ contains one free coordinate. Table 1.4 contains the conversions between screw transformations and homogeneous transformations. Note that the equivalent rotation matrix for a screw transformation has the same form as the equivalent rotation matrix for an angle-axis representation of orientation in Table 1.1. Also, the auxiliary relationship that the vector $\boldsymbol{\rho}$ be orthogonal to the screw axis is used in Table 1.4 to provide a unique conversion to the screw transformation.

Additional information on screw theory can be found in [2][11][7][3].

1.2.5 Plücker Coordinates

A minimum of four coordinates are needed to define a line in space. The Plücker coordinates of a line form a six-dimensional vector, so they are superabundant by two. They can be viewed as a pair of three-dimensional vectors; one is parallel to the line, and the other is the "moment" of that vector about the origin. Thus, if $\boldsymbol{u}$ is any vector parallel to the line and $\boldsymbol{\rho}$ is the position of any point on the line relative to the origin, the Plücker coordinates (L, M, N, P, Q, R) are given by:

$$
(L, M, N) = \boldsymbol{u}^T;\ (P, Q, R) = (\boldsymbol{\rho} \times \boldsymbol{u})^T. \tag{1.7}
$$

For simply defining a line, the magnitude of $\boldsymbol{u}$ is not unique, nor is the component of $\boldsymbol{\rho}$ parallel to $\boldsymbol{u}$. Two auxiliary relationships are imposed to reduce the set to just four independent coordinates. One is that the scalar product of the two three-dimensional vectors is identically zero.

$$
LP + MQ + NR \equiv 0. \tag{1.8}
$$

The other is the invariance of the line designated when the coordinates are all multiplied by the same scaling factor.

$$
(L, M, N, P, Q, R) \equiv (kL, kM, kN, kP, kQ, kR). \tag{1.9}
$$

This relationship may take the form of constraining $\boldsymbol{u}$ to have unit magnitude so that L, M, and N are the direction cosines.

In this handbook, it is often useful to express velocities in Plücker coordinates, wherein unlike the definition of lines, the magnitudes of the the two three-dimensional vectors are not arbitrary. Using the notation above, if $\boldsymbol{u}$ is the vector of a body's angular velocity $\boldsymbol{\omega}$ and $\boldsymbol{\rho}$ is the position vector from a point O fixed in the body to any point on the axis of rotation defined by $\boldsymbol{\omega}$, the Plücker coordinates are the angular velocity of the body and the translational velocity of the point O in the body. When the point O is the origin of the coordinate frame associated with that body, these coordinates are known as the spatial velocity of the body.

$$
\mathbf{v} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{\rho} \times \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{bmatrix}. \tag{1.10}
$$

1.3 Joint Kinematics

A kinematic joint is a connection between two bodies that constrains their relative motion. Two bodies that are in contact with one another create a simple kinematic joint. The surfaces of the two bodies that are in contact are able to move over one another, thereby permitting relative motion of the two bodies. Simple kinematic joints are classified as lower pair joints if contact occurs over surfaces and as higher pair joints if contact occurs only at points or along lines. Lower pair joints are mechanically attractive since wear is spread over the whole surface and lubricant is trapped in the small clearance space (in non-idealized systems) between the surfaces, resulting in relatively good lubrication. As can be proved from the requirement for surface contact [Ref], there are only six possible forms of lower pair: revolute, prismatic, helical, cylindrical, spherical, and planar joints.

Some higher pair joints also have attractive properties, particularly rolling pairs in which one body rolls without slipping over the surface of the other. This is mechanically attractive since the absence of sliding means the absence of abrasive wear. However, since contact occurs at a point, or along a line, application of a load across the joint may lead to very high local stresses resulting in other forms of material failure and, hence, wear. Higher pair joints can be used to create kinematic joints with special geometric properties, as in the case of a gear pair, or a cam and follower pair.

Compound kinematic joints are connections between two bodies formed by chains of other members and simple kinematic joints. A compound joint may constrain the relative motion of the two bodies joined in the same way as a simple joint. In such a case, the two joints are said to be kinematically equivalent. An example is a ball bearing that provides the same constraint as a revolute joint but is composed of a set of bearing balls trapped between two journals. The balls ideally roll without slipping on the journals, thereby taking advantage of the special properties of rolling contact joints.

Additional information on joints can be found in Chapter 4 Mechanisms and Actuation.

1.3.1 Revolute

The most general form of a revolute joint, sometimes referred to colloquially as a hinge or pin joint, is a lower pair composed of two congruent surfaces of revolution. The surfaces are the same except one of them is an external surface, convex in any plane normal to the axis of revolution, and one is an internal surface, concave in any plane normal to the axis. The surfaces may not be solely in the form of right circular cylinders, since surfaces of that form do not provide any constraint on axial sliding. A revolute joint permits only rotation of one of the bodies joined relative to the other. The position of one body relative to the other may be expressed as the angle between two lines normal to the joint axis, one fixed in each body. Thus, the joint has one degree of freedom. As noted above, a revolute joint is often realized as a compound joint. Revolute joints are easily actuated by rotating motors and are, therefore, extremely common in robotic systems. They may also be present as passive, unactuated joints.

When the $\hat{\boldsymbol{z}}$ axis of coordinate frame i is aligned with a revolute joint axis, the free modes $\boldsymbol{\Phi}_i$ and the constrained modes $\boldsymbol{\Phi}_i^c$ of the joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\Phi}_i^c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.11}$$

1.3.2 Prismatic

The most general form of a prismatic joint, sometimes referred to colloquially as a sliding joint, is a lower pair formed from two congruent general cylindrical surfaces. These may not be right circular cylindrical surfaces. A general cylindrical surface is obtained by extruding any curve in a constant direction. Again, one surface is internal and the other is an external surface.

A prismatic joint permits only sliding of one of the members joined relative to the other along the direction of extrusion. The position of one body relative to the other is determined by the distance between two points on a line parallel to the direction of sliding, with one point fixed in each body. Thus, this joint also has one degree of freedom.

A prismatic joint may be realized as a compound joint by means of a roller-rail assembly or other kinematically equivalent mechanism. Prismatic joints are also relatively easily actuated by means of linear actuators such as hydraulic or pneumatic cylinders, ball screws, or screw jacks. However, they always have motion limits since unidirectional sliding can, in principle, produce infinite displacements. They are commonly used in robotic mechanisms, but not nearly as commonly as revolutes.

When the $\hat{\boldsymbol{z}}$ axis of coordinate frame i is aligned with a prismatic joint axis, the free and constrained modes of the joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; \boldsymbol{\Phi}_i^c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.12}$$

1.3.3 Helical

The most general form of a helical joint, sometimes referred to colloquially as a screw joint, is a lower pair formed from two helicoidal surfaces formed by extruding any curve along a helical path. The simple example is a screw and nut wherein the basic generating curve is a pair of straight lines. The angle of rotation about the axis of the screw joint θ is directly related to the distance of displacement of one body relative to the other along that axis d by the expression $d = h\theta$, where the constant h is called the pitch of the screw. Screw joints are most often found in robotic mechanisms as constituents of linear actuators such as screw jacks and ball screws. They are seldom used as primary kinematic joints.

When the $\hat{\boldsymbol{z}}$ axis of coordinate frame i is aligned with a helical joint axis and assuming that the joint cannot be backdriven, the free and constrained modes of the joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ h \end{bmatrix}, \; \boldsymbol{\Phi}_i^c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.13}$$

1.3.4 Cylindrical

A cylindrical joint is a lower pair formed by contact of two congruent right circular cylinders, one an internal surface and the other an external surface. It permits both rotation about the cylinder axis and sliding parallel to it. Therefore, it is a joint with two degrees of freedom. Joints with more than one degree of freedom are only used passively in robotic mechanisms because each degree of freedom of an active joint must be separately actuated. However, the lower pair joints with more than one degree of freedom are easily replaced by kinematically equivalent compound joints that are serial chains of one degree of freedom lower pairs. In the present case, the cylindrical joint can be replaced by a revolute in series with a prismatic joint whose direction of sliding is parallel to the revolute axis. While simpler to implement using the geometric representation discussed in Section 1.5, this approach has disadvantages for dynamic simulation. Modeling a single cylindrical joint as a combination of a prismatic and revolute joint requires the addition of a virtual link between the two with zero mass and zero length. The massless link can create computational problems.

When the $\hat{\boldsymbol{z}}$ axis of coordinate frame i is aligned with a cylindrical joint axis, the free and constrained modes of the joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \boldsymbol{\Phi}_i^c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.14}$$

1.3.5 Spherical

A spherical joint is a lower pair formed by contact of two congruent spherical surfaces. Once again, one is an internal surface, and the other is an external surface. A spherical joint permits rotation about any line through the center of the sphere. Thus, it permits independent rotation about axes in up to three different directions and has three degrees of freedom. Although passive spherical joints are quite often found in robotic mechanisms, actuated spherical joints are not found. However, a spherical joint is easily replaced by a kinematically equivalent compound joint consisting of three revolutes that have concurrent axes. They do not need to be successively orthogonal, but often they are implemented that way. Each of the three revolute joints can be actuated. The arrangement is, in general, kinematically equivalent to a spherical joint, but it does exhibit a singularity when the revolute joint axes become coplanar. This is as compared to the native spherical joint that never has such a singularity. Likewise, if a spherical joint is modeled in simulation as three revolutes, computational difficulties again can arise from the necessary inclusion of massless virtual links having zero length.

The free and constrained modes of a spherical joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Phi}_i^c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.15}$$

1.3.6 Planar

A planar joint is formed by planar contacting surfaces. Like the spherical joint, it is a lower pair joint with three degrees of freedom. Passive planar joints are occasionally found in robotic mechanisms. A kinematically equivalent compound joint consisting of a serial chain of three revolutes with parallel axes can replace a planar joint. As was the case with the spherical joint, the compound joint exhibits a singularity when the revolute axes become coplanar.

When the $\hat{\boldsymbol{z}}$ axis of coordinate frame i is aligned with the normal to the plane of contact, the free and constrained modes of the joint are expressed as,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Phi}_i^c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.16}$$

1.3.7 Universal

A universal joint, often referred to as a Cardan or Hooke joint, is a compound joint with two degrees of freedom. It consists of a serial chain of two revolutes whose axes intersect orthogonally and is used in robotic mechanisms in both active and passive forms.

1.3.8 Rolling Contact

Rolling contact actually encompasses several different geometries. Planar rolling contact requires that the mechanism in which the joint is used constrain the relative motion to a plane. This is the case of a roller bearing, for example. Rolling contact in planar motion permits one degree of freedom of relative motion. As was noted above, rolling contact has desirable wear properties since the absence of sliding means the absence of abrasive wear. Planar rolling contact can take place along a line, thereby spreading the load and wear somewhat. Three-dimensional rolling contact allows rotation about any axis through the point of contact that is, in principle, unique. Hence, a three-dimensional rolling contact pair permits relative motion with three degrees of freedom.

1.3.9 Holonomic and Non-Holonomic Constraints

With the exception of rolling contact, all of the constraints associated with the joints discussed in the preceding sections can be expressed mathematically by equations containing only the joint position variables. These are called holonomic constraints. The number of equations, and hence the number of constraints, is $6 - n$, where n is the number of degrees of freedom of the joint. The constraints are intrinsically part of the axial joint model.

A non-holonomic constraint is one that cannot be expressed in terms of the position variables alone, but includes the time derivative of one or more of those variables. These constraint equations cannot be integrated to obtain relationships solely between the joint variables. The most common example in robotic systems arises from the use of a wheel or roller that rolls without slipping on another member.

1.3.10 Generalized Coordinates

In a robot manipulator consisting of N bodies, $6N$ coordinates are required to specify the position and orientation of all the bodies relative to a coordinate frame. Since some of those bodies are jointed together, a number of constraint equations will establish relationships between some of these coordinates. In this case, the $6N$ coordinates can be expressed as functions of a smaller set of coordinates $\boldsymbol{q}$ that are all independent. The coordinates in this set are known as generalized coordinates, and motions associated with these coordinates are consistent with all of the constraints. The joint variables $\boldsymbol{q}$ of a robot manipulator form a set of generalized coordinates.

1.4 Workspace

Most generally, the workspace of a robotic manipulator is the total volume swept out by the end-effector as the manipulator executes all possible motions. The workspace is determined by the geometry of the manipulator and the limits of the joint motions. It is more specific to define the reachable workspace as the total locus of points at which the end-effector can be placed and the dextrous workspace as the subset of those points at which the end-effector can be placed while having an arbitrary orientation. Dexterous workspaces exist only for certain idealized geometries, so real industrial manipulators with joint motion limits almost never possess dexterous workspaces.

Many serial-chain robotic manipulators are designed such that their joints can be partitioned into a regional structure and an orientation structure. The joints in the regional structure accomplish the positioning of the end-effector in space, and the joints in the orientation structure accomplish the orientation of the end-effector. Typically, the inboard joints of a serial chain manipulator comprise the regional structure, while the outboard joints comprise the orientation structure. Also, since prismatic joints provide no capability for rotation, they are generally not employed within the orientation structure.

The regional workspace volume can be calculated from the known geometry of the serial-chain manipulator and motion limits of the joints. With three inboard joints comprising the regional structure, the area of workspace for the outer two (joints 2 and 3) is computed first, and then the volume is calculated by integrating over the joint variable of the remaining inboard joint (joint 1). In the case of a prismatic joint, this simply involves multiplying the area by the total length of travel of the prismatic joint. In the more common case of a revolute joint, it involves rotating the area about the joint axis through the full range of motion of the revolute. By the Theorem of Pappus, the associated volume is equal to the product of the area, the distance from the area's centroid to the axis, and the angle through which the area is rotated. The boundaries of the area are determined by tracing the motion of a reference point in the end-effector, typically the center of rotation of the wrist that serves as the orientation structure. Starting with each of the two joints at motion limits and with joint 2 locked, joint 3 is moved until its second motion limit is reached. Joint 3 is then locked, and joint 2 is freed to move to its second motion limit. Joint 2 is again locked, while joint 3 is freed to move back to its original motion limit. Finally, joint 3 is locked, and joint 2 freed to move likewise to its original motion limit. In this way, the trace of the reference point is a closed curve whose area and centroid can be calculated mathematically.

More details on manipulator workspace can be found in Chapter 4 Mechanisms and Actuation and Chapter 11 Design Criteria.

1.5 Geometric Representation

The geometry of a robotic manipulator is conveniently defined by attaching coordinate frames to each link. While these frames could be located arbitrarily, it is advantageous both for consistency and computational efficiency to adhere to a convention for locating the frames on the links. Denavit and Hartenberg [8] introduced the foundational convention that has been adapted in a number of different ways, one of which is the convention introduced by Khalil and Dombre [12] used throughout this handbook. In all of its forms, the convention requires only four rather than six parameters to locate one coordinate frame relative to another. The four parameters are the link length a_i, the link twist α_i, the joint offset d_i, and the joint angle θ_i. This parsimony is achieved through judicious placement of the coordinate frame origins and axes such that the $\hat{\boldsymbol{x}}$ axis of one frame both intersects and is perpendicular to the $\hat{\boldsymbol{z}}$ axis of the preceding coordinate frame. The convention is applicable to manipulators consisting of revolute and prismatic joints, so when multi-degree-of-freedom joints are present, they are modeled as combinations of revolute and prismatic joints, as discussed in Section 1.3.

There are essentially four different forms of the convention for locating coordinate frames in a robotic mechanism. Each exhibits its own advantages by managing trade-offs of intuitive presentation. In the original Denavit and Hartenberg [8] convention, joint i is located between links i and $i+1$, so it is on the outboard side of link i. Also, the joint offset d_i and joint angle θ_i are measured along and about the $i-1$ joint axis, so the subscripts of the joint parameters do not match that of the joint axis. Waldron [35] and Paul [22] modified the labeling of axes in the original convention such that joint i is located between links $i-1$ and i in order to make it consistent with the base member of a serial chain being member 0. This places joint i at the inboard side of link i and is the convention used in all of the other modified versions. Furthermore, Waldron and Paul addressed the mismatch between subscripts of the joint parameters and joint axes by placing the $\hat{\boldsymbol{z}}_i$ axis along the $i+1$ joint axis. This, of course, relocates the subscript mismatch to the correspondence between the joint axis and the $\hat{\boldsymbol{z}}$ axis of the coordinate frame. Craig [6] eliminated all of the subscript mismatches by placing the $\hat{\boldsymbol{z}}_i$ axis along joint i, but at the expense of the homogeneous transform ${}^{i-1}\boldsymbol{T}_i$ being formed with a mixture of joint parameters with subscript i and link parameters with subscript $i-1$. Khalil and Dombre [12] introduced another

Geometric Parameters
Diagram

Figure 1.1: place holder

variation similar to Craig's except that it defines the link parameters a_i and α_i along and about the $\hat{\boldsymbol{x}}_{i-1}$ axis. In this case, the homogeneous transform ${}^{i-1}\boldsymbol{T}_i$ is formed only with parameters with subscript i, and the subscript mismatch is such that a_i and α_i indicate the length and twist of link $i-1$ rather than link i. Thus, in summary, the advantages of the convention used throughout this handbook compared to the alternative conventions are that the $\hat{\boldsymbol{z}}$ axes of the coordinate frames share the common subscript of the joint axes, and the four parameters that define the spatial transform from coordinate frame i to coordinate frame $i-1$ all share the common subscript i.

In this handbook, the convention for serial chain manipulators is shown in Figure 1.1 and summarized as follows:

1. The N moving bodies of the robotic mechanism are numbered from 1 to N. The number of the base is 0.

2. The N joints of the robotic mechanism are numbered from 1 to N, with joint i located between members $i-1$ and i.

3. The $\hat{\boldsymbol{z}}_i$ axis is located along the axis of joint i.

4. The $\hat{\boldsymbol{x}}_i$ axis is located along the common normal between the $\hat{\boldsymbol{z}}_{i-1}$ and $\hat{\boldsymbol{z}}_i$ axes.

5. a_i is the distance from $\hat{\boldsymbol{z}}_{i-1}$ to $\hat{\boldsymbol{z}}_i$ along $\hat{\boldsymbol{x}}_{i-1}$.

6. α_i is the angle from $\hat{\boldsymbol{z}}_{i-1}$ to $\hat{\boldsymbol{z}}_i$ about $\hat{\boldsymbol{x}}_{i-1}$.

7. d_i is the distance from $\hat{\boldsymbol{x}}_{i-1}$ to $\hat{\boldsymbol{x}}_i$ along $\hat{\boldsymbol{z}}_i$.

8. θ_i is the angle from $\hat{\boldsymbol{x}}_{i-1}$ to $\hat{\boldsymbol{x}}_i$ about $\hat{\boldsymbol{z}}_i$.

The geometric parameters for the example manipulator shown in Figure 1.2 are listed in Table 1.5. All of the joints of this manipulator are revolutes, and joint 1 has a vertical orientation. Joint 2 is perpendicular to joint 1 and intersects it. Joint 3 is parallel to joint 2, and the length of link 2 is a_3. Joint 4 is perpendicular to joint 3 and intersects it. Joint 5 likewise intersects joint 4 perpendicularly at an offset of d_4 from joint 3. Finally, joint 6 intersects joint 5 perpendicularly.

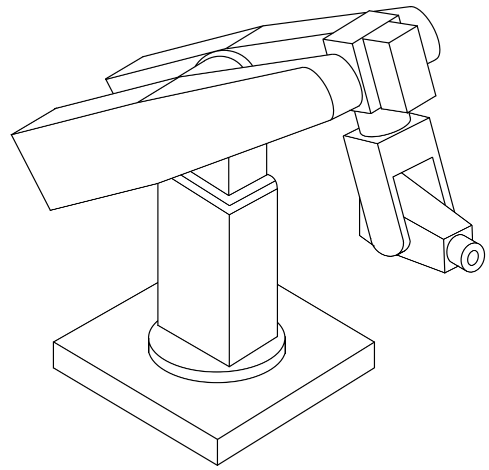

Figure 1.2: Example Six-Degree-of-Freedom Serial Chain Manipulator.

i	α_i	a_i	d_i	θ_i
1	0	0	0	θ_1
2	$-\frac{\pi}{2}$	0	0	θ_2
3	0	a_3	0	θ_3
4	$-\frac{\pi}{2}$	0	d_4	θ_4
5	$\frac{\pi}{2}$	0	0	θ_5
6	$-\frac{\pi}{2}$	0	0	θ_6

Table 1.5: Tabulated Geometric Parameters for Example Serial Chain Manipulator.

With this convention, coordinate frame i can be located relative to coordinate frame $i-1$ by executing a rotation through an angle α_i about the $\hat{\boldsymbol{x}}_{i-1}$ axis, a translation of distance a_i along $\hat{\boldsymbol{x}}_{i-1}$, a rotation through an angle θ_i about the $\hat{\boldsymbol{z}}_i$ axis, and a translation of distance d_i along $\hat{\boldsymbol{z}}_i$. Through concatentation of these individual transformations, the equivalent homogeneous transformation is,

$$
{}^{j}\boldsymbol{T}_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & a_i \\ s_{\theta_i}c_{\alpha_i} & c_{\theta_i}c_{\alpha_i} & -s_{\alpha_i} & -s_{\alpha_i}d_i \\ s_{\theta_i}s_{\alpha_i} & c_{\theta_i}s_{\alpha_i} & c_{\alpha_i} & c_{\alpha_i}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (1.17)
$$

The identification of geometric parameters is addressed in Chapter 3 Model Identification.

1.6 Forward Kinematics

The forward kinematics problem for a serial chain manipulator is to find the position and orientation of the end-effector relative to the base given the positions of all of the joints and the values of all of the geometric link parameters. A more general expression of the problem is to find the relative position and orientation of any two designated members given the geometric structure of the manipulator and the values of a number of joint positions equal to the number of degrees of freedom of the mechanism. The forward kinematics problem is critical for developing manipulator coordination algorithms because joint positions are typically measured by sensors mounted on the joints and it is necessary to calculate the positions of the joint axes relative to the fixed reference frame.

In practice, the forward kinematics problem is solved by calculating the transformation between a coordinate frame fixed in the end-effector and another coordinate frame fixed in the base. This is straightforward for a serial chain since the transformation describing the position of the end-effector relative to the base is obtained by simply concatenating transformations between frames fixed in adjacent links of the chain. The convention for the geometric representation of a manipulator presented in Section 1.5 reduces this to finding an equivalent 4×4 homogeneous transformation matrix that relates the spatial displacement of the end-effector coordinate frame to the base coordinate frame.

For the example serial chain manipulator shown in Figure 1.2, the transformation is,

$$ {}^0\boldsymbol{T}_6 = {}^0\boldsymbol{T}_1 {}^1\boldsymbol{T}_2 {}^2\boldsymbol{T}_3 {}^3\boldsymbol{T}_4 {}^4\boldsymbol{T}_5 {}^5\boldsymbol{T}_6. \qquad (1.18) $$

Table 1.6 contains the elements of ${}^0\boldsymbol{T}_6$ that are calculated using Table 1.5 and Equation 1.17.

Once again, homogeneous transformations provide a compact notation, but are computationally inefficient for solving the forward kinematics problem. A reduction in computation can be achieved by separating the position and orientation portions of the transformation to eliminate all multiplications by the 0 and 1 elements of the matrices.

The forward kinematics problem for closed chains is much more complicated because of the additional constraints present. Solution methods for closed chains are included in Chapter 13 Kinematic Structure and Analysis.

$$
\begin{aligned}
{}^0\boldsymbol{T}_6 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^0p_6^x \\ r_{21} & r_{22} & r_{23} & {}^0p_6^y \\ r_{31} & r_{32} & r_{33} & {}^0p_6^z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
r_{11} &= c_1 (s_2 s_3 - c_2 c_3)(s_4 s_6 - c_4 c_5 c_6) \\
&\quad -c_1 s_5 c_6 (c_2 s_3 + s_2 c_3) + s_1 (s_4 c_5 c_6 + c_4 s_6). \\
r_{21} &= s_1 (s_2 s_3 - c_2 c_3)(s_4 s_6 - c_4 c_5 c_6) \\
&\quad -s_1 s_5 c_6 (c_2 s_3 + s_2 c_3) - c_1 (s_4 c_5 c_6 + c_4 s_6). \\
r_{31} &= (c_2 s_3 + s_2 c_3)(s_4 s_6 - c_4 c_5 c_6) \\
&\quad +s_5 c_6 (s_2 s_3 - c_2 c_3). \\
r_{12} &= c_1 (s_2 s_3 - c_2 c_3)(c_4 c_5 s_6 + s_4 c_6) \\
&\quad +c_1 s_5 s_6 (c_2 s_3 + s_2 c_3) + s_1 (c_4 c_6 - s_4 c_5 s_6). \\
r_{22} &= s_1 (s_2 s_3 - c_2 c_3)(c_4 c_5 s_6 + s_4 c_6) \\
&\quad +s_1 s_5 s_6 (c_2 s_3 + s_2 c_3) - c_1 (c_4 c_6 - s_4 c_5 s_6). \\
r_{32} &= (c_2 s_3 + s_2 c_3)(c_4 c_5 s_6 + s_4 c_6) \\
&\quad -s_5 s_6 (s_2 s_3 - c_2 c_3). \\
r_{13} &= c_1 c_4 s_5 (s_2 s_3 - c_2 c_3) - c_1 c_5 (c_2 s_3 + s_2 c_3) \\
&\quad -s_1 s_4 s_5. \\
r_{23} &= s_1 c_4 s_5 (s_2 s_3 - c_2 c_3) - s_1 c_5 (c_2 s_3 + s_2 c_3) \\
&\quad +c_1 s_4 s_5. \\
r_{33} &= c_4 s_5 (c_2 s_3 + s_2 c_3) + c_5 (s_2 s_3 - c_2 c_3). \\
{}^0p_6^x &= a_2 c_1 c_2 - d_4 c_1 (c_2 s_3 + s_2 c_3) \\
{}^0p_6^y &= a_2 s_1 c_2 - d_4 s_1 (c_2 s_3 + s_2 c_3) \\
{}^0p_6^z &= -a_2 s_2 + d_4 (s_2 s_3 - c_2 c_3)
\end{aligned}
$$

Table 1.6: Forward Kinematics of the Example Serial Chain Manipulator in Figure 1.2.

1.7 Inverse Kinematics

The inverse kinematics problem for a serial chain manipulator is to find the values of the joint positions given the position and orientation of the end-effector relative to the base and the values of all of the geometric link parameters. Once again, this is a simplified statement applying only to serial chains. A more general statement is: Given the relative positions and orientations of two members of a mechanism, find the values of all of the joint positions. This amounts to finding all of the joint positions given the homogeneous transformation between the two members of interest. In the common case of a six-degree-of-freedom serial chain manipulator, the known transformation is ${}^0\boldsymbol{T}_6$. Reviewing the formulation of this transformation in Section 1.6, it is clear that the inverse kinematics problem for serial chain manipulators requires the solution of non-linear sets of equations. In the case of a six-degree-of-freedom manipulator, three of these equations relate to the position vector within the homogeneous transform, and the other three relate to the rotation matrix. In the latter case, these three equations cannot come from the same row or column because of the dependency within the rotation matrix. With these non-linear equations, it is possible that no solutions exist or multiple solutions exist. For a solution to exist, the desired position and orientation of the end-effector must lie in the workspace of the manipulator. In cases where solutions do exist, they often cannot be presented in closed form, so numerical methods are required.

1.7.1 Closed-Form Solutions

Closed-form solutions are desirable because they are faster than numerical solutions and readily identify all possible solutions. The disadvantage of closed-form solutions are that they are not general, but robot-dependent. The most effective methods for finding closed-form solutions are ad hoc techniques that take advantage of particular geometric features of specific mechanisms. In general, closed-form solutions can only be obtained for six-degree-of-freedom systems with special kinematic structure characterized by a large number of the geometric parameters defined in Section 1.5 being zero-valued. Most industrial manipulators have such structure because it permits more efficient coordination software. Sufficient conditions for a six-degree-of-freedom manipulator to have closed-form inverse kinematics solutions are [26]: 1) three consecutive revolute joint axes intersect at a common point, as in a spherical wrist; 2) three consecutive revolute joint axes are parallel.

Closed-form solution approaches are generally divided into algebraic and geometric methods.

Algebraic Methods

Algebraic methods involve identifying the significant equations containing the joint variables and manipulating them into a soluble form. A common strategy is reduction to a transcendental equation in a single variable such as,

$$C_1 \cos\theta_i + C_2 \sin\theta_i + C_3 = 0, \tag{1.19}$$

where C_1, C_2, and C_3 are constants. The solution to such an equation is,

$$\theta_i = 2\tan^{-1}\left(\frac{C_2 \pm \sqrt{C_2^2 - C_3^2 + C_1^2}}{C_1 - C_3}\right). \tag{1.20}$$

Special cases in which one or more of the constants are zero are also common. Reduction to a pair of equations having the form,

$$C_1 \cos\theta_i + C_2 \sin\theta_i + C_3 = 0$$

$\theta_1 =$
$\theta_3 =$
$\theta_2 =$

Table 1.7: Inverse Position Kinematics of the Articulated Arm Within the Example Serial Chain Manipulator in Figure 1.2.

$\theta_4 =$
$\theta_5 =$
$\theta_6 =$

Table 1.8: Inverse Orientation Kinematics of the Spherical Wrist Within the Example Serial Chain Manipulator in Figure 1.2.

$$C_1 \sin\theta_i - C_2 \cos\theta_i + C_4 = 0, \tag{1.21}$$

is particularly useful because only one solution results,

$$\theta_i = Atan2\left(-C_1C_4 - C_2C_3, C_2C_4 - C_1C_3\right). \tag{1.22}$$

Geometric Methods

Geometric methods involve identifying points on the manipulator relative to which position and/or orientation can be expressed as a function of a reduced set of the joint variables. This often amounts to decomposing the spatial problem into separate planar problems. The resulting equations are solved using algebraic manipulation. The two sufficient conditions for existence of a closed-form solution for a six-degree-of-freedom manipulator that are listed above enable the decomposition of the problem into inverse position kinematics and inverse orientation kinematics. This is the decomposition into regional and orientation structures discussed in Section 1.4, and the solution is found by rewriting Equation 1.18,

(1.23)

The example manipulator in Figure 1.2 has this structure, and regional structure is commonly known as an articulated or anthropomorphic arm or an elbow manipulator. The solution to the inverse position kinematics problem for such a structure is summarized in Table 1.7.1. The orientation structure is simply a spherical wrist, and the corresponding solution to the inverse orientation kinematics problem is summarized in Table 1.7.1.

1.7.2 Numerical Methods

Unlike the algebraic and geometric methods used to find closed-form solutions, numerical methods are not robot-dependent, so they can be applied to any kinematic structure. The disadvantages of numerical methods are that they can be slower and in some cases, they do not allow computation of all possible solutions. For a six-degree-of-freedom serial chain manipulator with only revolute and prismatic joints, the translation and rotation equations can always be reduced to a polynomial in a single variable of degree not greater than 16 [15]. Thus, such a manipulator can have as many as sixteen real solutions to the inverse kinematics problem [18]. Since closed form solution of a polynomial equation is only possible if the polynomial is of degree four or less, it follows that many manipulator geometries are not soluble in closed form. In general, a greater number of non-zero geometric parameters corresponds to a polynomial of higher degree in the reduction. For such manipulator structures, the most common numerical methods can be divided into categories of symbolic elimination methods, continuation methods, and interative methods.

Symbolic Elimination Methods

Symbolic elimination methods involve analytical manipulations to eliminate variables from the system of nonlinear equations to reduce it to a smaller set of equations. Raghavan and Roth [27] used dialytic elimination to reduce the inverse kinematics problem of a general six-revolute serial chain manipulator to a polynomial of degree 16 and to find all possible solutions. The roots provide solutions for one of the joint variables, while the other variables are computed by solving linear systems. Manocha and Canny [17] improved the numerical properties of this technique by reformulating the problem as a generalized eigenvalue problem. An alternative approach to elimination makes use of Gröbner bases [4][13].

Continuation Methods

Continuation methods involve tracking a solution path from a start system with known solutions to a target system whose solutions are sought as the start system is transformed into the target system. These techniques have been applied to inverse kinematics problems [33], and special properties of polynomial systems can be exploited to find all possible solutions [36].

Iterative Methods

A number of different iterative methods can be employed to solve the inverse kinematics problem. Most of them converge to a single solution based on an initial guess, so the quality of that guess greatly impacts the solution time. Newton-Raphson methods provide a fundamental approach that uses a first-order approximation to the original equations. Pieper [26] was among the first to apply the method to inverse kinematics, and others have followed [32][19]. Optimization approaches formulate the problem as a nonlinear optimization problem and employ search techniques to move from an initial guess to a solution [34][40]. Resolved motion rate control converts the problem to a differential equation [37], and a modified predictor-corrector algorithm can be used to perform the joint velocity integration [5]. Control-theory-based methods cast the differential equation into a control problem [29]. Interval analysis [28] is perhaps one of the most promising iterative methods because it offers rapid convergence to a solution and can be used to find all possible solutions.

1.8 Forward Instantaneous Kinematics

The forward instantaneous kinematics problem for a serial chain manipulator is: Given the positions of all members of the chain and the rates of motion about all the joints, find the total velocity of the end-effector. Here the rate of motion about the joint is the angular velocity of rotation about a revolute joint or the translational velocity of sliding along a prismatic joint. The total velocity of a member is the velocity of the origin of the reference frame fixed to it combined with its angular velocity. That is, the total velocity has six independent components and therefore, completely represents the velocity field of the member. It is important to notice that this definition includes an assumption that the position of the mechanism is completely known. In most situations, this means that either the forward or inverse position kinematics problem must be solved before the forward instantaneous kinematics problem can be addressed. The same is true of the inverse instantaneous kinematics problem discussed in the following section. The forward instantaneous kinematics problem is important when doing acceleration analysis for the purpose of studying dynamics. The total velocities of the members are needed for the computation of Coriolis and centripetal acceleration components.

Differentiation with respect to time of the forward position kinematics equations yields a set of equations of the form,

$$\mathbf{v}_N = \boldsymbol{J}(\mathbf{q})\dot{\mathbf{q}} \tag{1.24}$$

where $\mathbf{v}_N$ is the spatial velocity of the end-effector, $\dot{\mathbf{q}}$ is an N-dimensional vector composed of the joint rates, and $\boldsymbol{J}(\mathbf{q})$ is a $6 \times N$ matrix whose elements are, in general, non-linear functions of $\mathbf{q}_1$, ..., $\mathbf{q}_N$. $\boldsymbol{J}(\mathbf{q})$ is called the Jacobian matrix of this algebraic system. If the joint positions are known, Equation 1.24 yields six linear algebraic equations in the joint rates. If the joint rates are given, solution of Equation 1.24 is a solution of the forward instantaneous kinematics problem. Notice that $\boldsymbol{J}(\mathbf{q})$ can be regarded as a known matrix for this purpose provided all the joint positions are known.

1.8.1 Jacobian

Formulation of the Jacobian. Differentiation of position equations. From screw parameters. Development of equation for transforming the Jacobian from one frame to another.

Table 1.8.1 contains the Jacobian of the example manipulator show in Figure 1.2.

Additional information about Jacobians can be found in Chapter 12 Kinematically Redundant Manipulators.

1.9 Inverse Instantaneous Kinematics

The important problem from the point of view of robotic coordination is the inverse instantaneous kinematics problem. The inverse instantaneous kinematics problem for a serial chain manipulator is: Given the positions of all members of the chain and the total velocity of the end-effector, find the rates of motion of all joints. When controlling a movement of an industrial robot which operates in the point-to-point mode, it is not only necessary to compute the final joint positions needed to assume the desired final hand position. It is also necessary to generate a smooth trajectory for motion between the initial and final positions. There are, of course, an infinite number of possible trajectories for this purpose. However, the most straightforward and successful approach employs algorithms based on the solution of the inverse instantaneous kinematics problem. This technique originated in the work of Whitney [38] and of Pieper [26].

$$\boldsymbol{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}$$

$J_{11} =$
$J_{21} =$
$J_{31} =$
$J_{41} =$
$J_{12} =$
$J_{22} =$
$J_{32} =$
$J_{42} =$
$J_{13} =$
$J_{23} =$
$J_{33} =$
$J_{43} =$
$J_{14} =$
$J_{24} =$
$J_{34} =$
$J_{44} =$

Table 1.9: Forward Kinematics of the Example Serial Chain Manipulator in Figure 1.2.

1.9.1 Inverse Jacobian

In order to solve the linear system of equations in the joint rates obtained by decomposing Equation 1.24 into its component equations when $\mathbf{v}$ is known, it is necessary to invert the Jacobian matrix. The equation becomes,

$$\dot{\mathbf{q}} = \boldsymbol{J}^{-1}(\mathbf{q})\mathbf{v}_N \tag{1.25}$$

Since $\boldsymbol{J}$ is a 6×6 matrix, numerical inversion is not very attractive in real-time software which must run at computation cycle rates of the order of 100 Hz or more. Worse, it is quite possible for $\boldsymbol{J}$ to become singular ($|\boldsymbol{J}| = 0$). The inverse does not then exist. Even when the Jacobian matrix does not become singular, it may become ill conditioned leading to degraded performance in significant portions of the manipulator's workspace. Most industrial robot geometries are simple enough that the Jacobian matrix can be inverted analytically leading to a set of explicit equations for the joint rates. This greatly reduces the number of computation operations needed as compared to numerical inversion.

More information on singularities can be found in Chapter 4 Mechanisms and Actuation and Chapter 13 Kinematic Structure and Analysis.

1.10 Static Wrench Transmission

Static wrench analysis of a manipulator establishes the relationship between wrenches applied to the end-effector and forces/torques applied to the joints. Through the priniciple of virtual work, this relationship can be shown to be,

$$\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{f}. \tag{1.26}$$

1.11 Conclusions and Further Reading

A number of excellent texts provide a broad introduction to robotics with significant focus on kinematics [1][6][12][16][22][29][30][31][39][20].

Bibliography

[1] H. Asada and J.-J. E. Slotine, *Robot Analysis and Control*, New York: John Wiley & Sons, 1986.

[2] R. S. Ball, *A Treatise on the Theory of Screws*, Cambridge: Cambridge University Press, 1998.

[3] O. Bottema and B. Roth, *Theoretical Kinematics*, New York: Dover Publications, 1990.

[4] B. Buchberger, "Applications of Grobner Basis in Non-Linear Computational Geometry," *Trends in Computer Algebra. Lecture Notes in Computer Science*, vol. 296, Springer Verlag, 1989.

[5] H. Cheng and K. Gupta, "A Study of Robot Inverse Kinematics Based Upon the Solution of Differential Equations," *Journal of Robotic Systems*, vol. 8, no. 2, pp. 115–175, 1991.

[6] J. J. Craig, *Introduction to Robotics: Mechanics and Control*, Reading, MA: Addison-Wesley, 1986.

[7] J. K. Davidson and K. H. Hunt, *Robots and Screw Theory: Applications of Kinematics and Statics to Robotics*, Oxford: Oxford University Press, 2004.

[8] J. Denavit and R. S. Hartenberg, "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," *Journal of Applied Mechanics*, vol. 22, pp. 215–221, 1955.

[9] J. Duffy, *Analysis of Mechanisms and Robot Manipulators*, New York: Wiley, 1980.

[10] K. S. Fu, R. C. Gonzalez, and C. S. G. Lee, *Robotics: Control, Sensing, Vision, and Intelligence*, New York: McGraw-Hill, 1987.

[11] K. H. Hunt, *Kinematic Geometry of Mechanisms*, Oxford: Clarendon Press, 1978.

[12] W. Khalil and E. Dombre, *Modeling, Identification and Control of Robots*, New York: Taylor & Francis, 2002.

[13] P. Kovacs, "Minimum Degree Solutions for the Inverse Kinematics Problem by Application of the Buchberger Algorithm," *Advances in Robot Kinematics*, S. Stifter and J. Lenarcic Eds., Springer Verlag, pp. 326–334, 1991.

[14] C. S. G Lee, "Robot Arm Kinematics, Dynamics, and Control," *Computer*, vol. 15, no. 12, pp. 62-80, 1982.

[15] H. Y. Lee and C. G. Liang, "A New Vector Theory for the Analysis of Spatial Mechanisms," *Mechanisms and Machine Theory*, vol. 23, no. 3, pp. 209–217, 1988.

[16] F. L. Lewis, C. T. Abdallah, and D. M. Dawson, *Control of Robot Manipulators*, New York: Macmillan, 1993.

[17] D. Manocha and J. Canny, *Real Time Inverse Kinematics for General 6R Manipulators*, Technical report, University of California, Berkeley, 1992.

[18] R. Manseur and K. L. Doty, "A Robot Manipulator with 16 Real Inverse Kinematic Solutions," *International Journal of Robotics Research*, vol. 8, no. 5, pp. 75–79, 1989.

[19] R. Manseur and K. L. Doty, "Fast Inverse Kinematics of 5-Revolute-Axis Robot Maniputlators," *Mechanisms and Machine Theory*, vol. 27, no. 5, pp. 587–597, 1992.

[20] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, Boca Raton, FL: CRC Press, 1994.

[21] D.E. Orin and W.W. Schrader, "Efficient Computation of the Jacobian for Robot Manipulators," *International Journal of Robotics Research*, vol. 3, no. 4, pp. 66–75, 1984.

[22] R. Paul, *Robot Manipulators: Mathematics, Programming and Control*, Cambridge, MA: MIT Press, 1982.

[23] R. P. Paul, B. E. Shimano, and G. Mayer, "Kinematic Control Equations for Simple Manipulators," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. SMC-11, no. 6, pp. 339-455, 1981.

[24] R. P. Paul and C. N. Stephenson, "Kinematics of Robot Wrists," *International Journal of Robotics Research*, vol. 20, no. 1, pp. 31–38, 1983.

[25] R. P. Paul and H. Zhang, "Computationally Efficient Kinematics for Manipulators with Spherical wrists based on the Homogeneous Transformation Representation," *International Journal of Robotics Research*, vol. 5, no. 2, pp. 32–44, 1986.

[26] D. Pieper, *The Kinematics of Manipulators Under Computer Control*, Doctoral Dissertation, Stanford University, 1968.

[27] M. Raghavan and B. Roth, "Kinematic Analysis of the 6R Manipulator of General Geometry," In *Proceedings of the 5th International Symposium on Robotics Research*, 1990.

[28] R. S. Rao, A. Asaithambi, and S. K. Agrawal, "Inverse Kinematic Solution of Robot Manipulators Using Interval Analysis," *ASME Journal of Mechanical Design*, vol. 120, no. 1, pp. 147–150, 1998.

[29] L. Sciavicco and B. Siciliano, *Modeling and Control of Robot Manipulators*, London: Springer, 2000.

[30] R. J. Schilling, *Fundamentals of Robotics: Analysis and Control*, Englewood Cliffs, NJ: Prentice-Hall, 1990.

[31] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*, New York: John Wiley & Sons, 1989.

[32] S. C. A. Thomopoulos, and R. Y. J. Tam, "An Iterative Solution to the Inverse Kinematics of Robotic Manipulators," *Mechanisms and Machine Theory*, vol. 26, no. 4, pp. 359–373, 1991.

[33] L. W. Tsai and A. P. Morgan "Solving the Kinematics of the Most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods," *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, vol. 107, pp. 189-195, 1985.

[34] J. J. Uicker, Jr., J. Denavit, and R. S. Hartenberg, "An Interactive Method for the Displacement Analysis of Spatial Mechanisms," *Journal of Applied Mechanics*, vol. 31, pp. 309–314, 1964.

[35] K. J. Waldron, "A Study of Overconstrained Linkage Geometry by Solution of Closure Equations, Part I: A Method of Study," *Mechanism and Machine Theory*, vol. 8, no. 1, pp. 95–104, 1973.

[36] C. W. Wampler, A. P. Morgan, and A. J. Sommese, "Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics," *ASME Journal of Mechanical Design*, vol. 112, pp. 59-68, 1990.

[37] D. E. Whitney, "Resolved Motion Rate Control of Manipulators and Human Prostheses," *IEEE Transactions on Man-Machine Systems*, vol. 10, pp. 47–63, 1969.

[38] D. E. Whitney, "The Mathematics of Coordinated Control of Prosthetic Arms and Manipulators," *Journal of Dynamic Systems, Measurement, and Control*, vol. 122, pp. 303–309, 1972.

[39] T. Yoshikawa, *Foundations of Robotics*, Cambridge, MA: MIT Press, 1990.

[40] J. Zhao and N. Badler, "Inverse Kinematics Positioning Using Nonlinear Programming for Highly Articulated Figures," *Transactions on Computer Graphics*, vol. 13, no. 4, pp. 313–336, 1994.