Collision Detection Between two Circular Rigid Bodies

A comparison of two methods: Periodic Interference Test (PIT) and Predicted Instance of Collision (PIC) Calculation

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Abstract: *Collision detection is an integral part of animating objects in a scene. When simulating real-world motions, a graphics designer needs to accurately mimic the effect of two rigid bodies colliding with each other. The laws of physics govern ideal (elastic) rigid body collisions, and these can be simulated in a graphics program. We explain and evaluate two methods of performing collision detection: Periodic Interference Test (PIT) and Predicted Instance of Collision (PIC) calculation.*

Periodic Interference Test (PIT)

In PIT, at every frame in the animation, there occurs a check to detect if any two discs are colliding. This check is done by testing if the discs are approaching each other, and if they interfere spatially. If a collision is detected, the new velocities after the shock are computed (using equations derived in a later section) and the discs move along the new path with the new velocity. If there is no collision in that frame, the animation moves along to the next frame. There occurs no check between two successive frames to detect a possible collision.

The PIT approach is a *lazy* collision detection strategy. While animating using PIT, the designer assumes that no collision occurs between frames. If a collision occurs, it will be detected in the next frame. If so, the velocities after collision are calculated then and the discs change course. This can be compared to Lazy Deadlock Detection in Operating Systems processes.

PIT is described by Algorithm 1.

```
PIT( Disc A, Disc B ):

At every frame instant t:
    Test if the two discs A and B are
    approaching each other.
    If YES:
        Test if they overlap (interfere)
        spatially.
        If YES:
                Collision detected.
                Compute New Velocities.
        Else:
                No collision detected.
    Advance to Next Frame instant.
```

Algorithm 1: PIT

Predicted Instant of Collision (PIC)

In PIC, the time to the earliest collision is calculated. If the time is not earlier than the next frame instant, the animation continues till the frame in which collision occurs. When collision occurs, the velocities of the bodies after the shock are calculated. The bodies then move along the new path with the new velocity. This method pre-calculates the exact time of collision, and does not allow the discs to interfere at all.

The PIC approach is a *proactive* collision detection strategy. The designer estimates the time of collision using the formulae obtained below, and thus predicts the precise moment of collision. The discs are then moved to that instant in time, wherever it occurs between any two frame instances. The velocities after collision are calculated precisely at this point of collision. Thus, the PIC approach models a collision perfectly, though at the cost of some extra computation. Thus, this is similar to Deadlock Prevention as compared to Detection and Recovery.

PIC is described in Algorithm 2.

```
PIC( Disc A, Disc B ):

At every frame instant t:
    Test if the two discs A and B are
    approaching each other.
    If YES:
        T:= time of collision between A and B.
        If T < next frame instant:
            Move frame to intermediate
            instant T.
            Calculate velocities after collision.
        Goto next frame instant.
    Else:
        Goto next frame instant.
```

Algorithm 2: PIC

The Physics and the Math of Collisions

Consider two rigid balls as shown in the Figure 1. The ball with center at C_1 has a radius r_1 and it moves with a constant velocity V_1. The ball with center at C_2 has a radius r_2 and moves with a constant velocity V_2. At time t, the centers of the two balls are at positions C_1' and C_2' and the two balls touch each other i.e. they collide.

The aim is:

1. To find the ***time t*** at which they collide.
2. To find their velocities *after* the collision.

In time t, the new co-ordinates of the centers are C_1' and C_2' respectively.

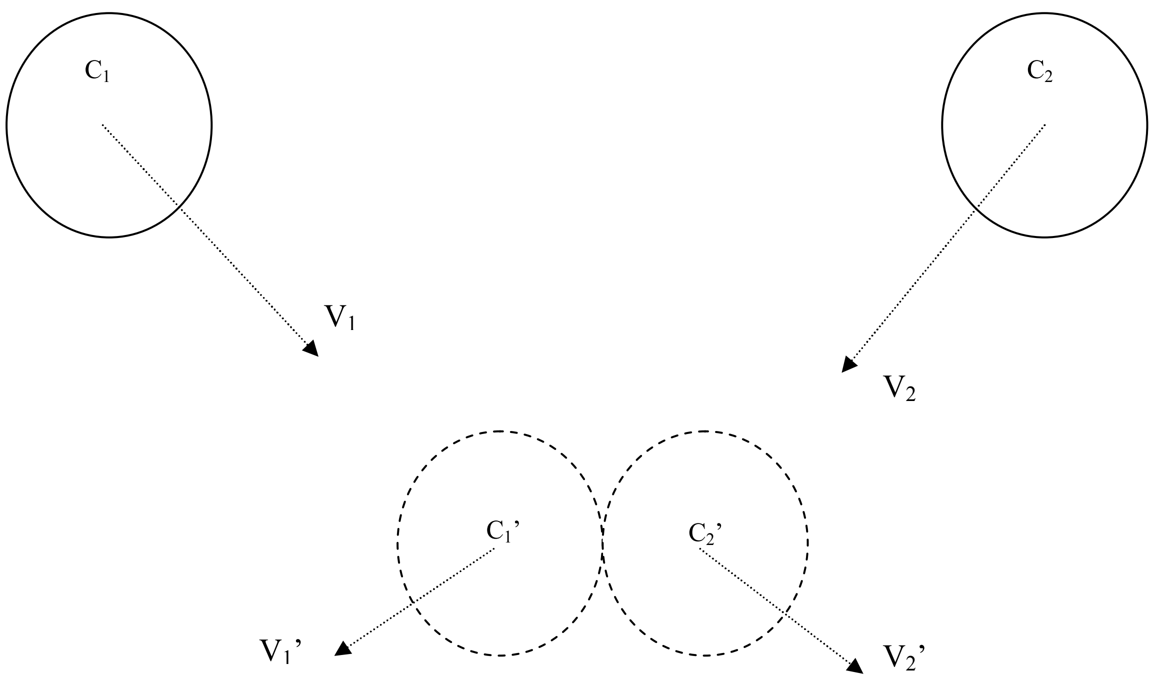


Figure 1: Collision between two rigid discs

Thus,

$C_1 + t\, V_1 = C_1'$ … (1)

$C_2 + t\, V_2 = C_2'$ … (2)

The distance between C_1' and C_2' at the time of collision is $(r_1 + r_2)$

Thus,

$\| C_1' C_2'\| = (r_1 + r_2)$

$\| C_2' - C_1'\| = (r_1 + r_2)$

Squaring both sides,

$\| C_2' - C_1'\|^2 = (r_1 + r_2)^2$

Substituting (1) and (2):

$((C_2 + t\, V_2) – (C_1 + t\, V_1))^2 = (r_1 + r_2)^2$

By property of vectors,

$V^2 = n(V)^2 = V.V$

$((C_2 + t\, V_2) – (C_1 + t\, V_1)) . ((C_2 + t\, V_2) – (C_1 + t\, V_1)) = (r_1 + r_2)^2$

$((C_2 - C_1) –t(V_2- V_1)). ((C_2 - C_1) –t(V_2 – V_1)) = (r_1 + r_2)^2$

$(C_1\, C_2 + t\, V_1\, V_2). (C_1\, C_2 + t\, V_1\, V_2) = (r_1 + r_2)^2$

$(C_1\, C_2). (C_1\, C_2) + t.(C_1\, C_2). (V_1\, V_2) + t. (V_1\, V_2) .(C_1\, C_2) + t .(V_1\, V_2). t.(V_1\, V_2) = (r_1 + r_2)^2$

$\| C_1\, C_2\|^2 + 2t.(C_1\, C_2). (V_1\, V_2) + t^2\| V_1\, V_2\|^2 = (r_1 + r_2)^2$

Rearranging the above equation,

$\| V_1\, V_2\|^2 .t^2 + 2.(C_1\, C_2). (V_1\, V_2)t + \| C_1\, C_2\|^2 - (r_1 + r_2)^2 = 0$ … (3)

Also, by the property of vectors,

$\|V\|^2 = V_x^2 + V_y^2$

Substituting this in (3):

$[(V_{2x} + V_{2y}) - (V_{1x} + V_{1y})]^2.t^2 + 2.(C_{2x} + C_{2y} – C_{1x} - C_{1y}).(V_{2x} + V_{2y} – V_{1x} - V_{1y})t + [(C_{2x} + C_{2y}) - (C_{1x} + C_{1y})]^2 – (r_1 + r_2)^2 = 0$

… (4)

For a quadratic equation of the form

$at^2 + bt + c = 0$,

the roots of the equation (i.e. values of t which satisfy the equation) are given by,

$t = (-b + sqrt(b^2 – 4ac))/2a$ root 1

$t = (-b - sqrt(b^2 – 4ac))/2a$ root 2

where,

$a = [(V_{2x} + V_{2y}) - (V_{1x} + V_{1y})]^2$

$b = 2.(C_{2x} + C_{2y}– C_{1x} - C_{1y}).(V_{2x} + V_{2y}– V_{1x} - V_{1y})$

$c = [(C_{2x} + C_{2y}) - (C_{1x} + C_{1y})]^2 – (r_1 + r_2)^2$

To find which of the two roots we should consider, we perform the following analysis:

1. Find the discriminant $b^2 – 4ac$.
   if $b^2 – 4ac < 0$ ignore the root
   else, calculate the root t_i and
   if $t_i \geq 0$, this is the time of collision.

2. If there are 2 such roots t_1 and t_2 after step 1, the collision happens at that value t_i such that,
   $t_i \geq 0$ and
   $t_i = \min (t_1 , t_2)$
   This t_i is the time of collision

To find the velocities V_1' and V_2' after the collision:

Let m_1 and m_2 be the masses of the balls with centers C_1 and C_2 respectively. By conservation of momentum, assuming a perfectly elastic collision:

Total momentum before collision = Total momentum after collision

Thus substituting the values,
$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2'$

Squaring this, we get
$m_1^2V_1^2 + m_2^2V_2^2 + 2\, m_1\, m_2\, V_1\, V_2 = m_1^2\, V_1'^2 + m_2^2V_2'^2 + 2\, m_1\, m_2\, V_1'\, V_2'$ … (5)

Also, by conservation of energy

Total kinetic energy before collision = Total kinetic energy after collision

Substituting the values,

$\frac{1}{2}\, m_1V_1^2 + \frac{1}{2}\, m_2V_2^2 = \frac{1}{2}\, m_1V_1'^2 + \frac{1}{2}\, m_2V_2'^2$

$m_1V_1^2 + m_2V_2^2 = m_1V_1'^2 + m_2V_2'^2$ … (6)

We solve (5) and (6), to get the values of V_1' and V_2'. To express these values elegantly, we use the concept of relative velocity:

Let N be the unit vector along the line passing through both the centers.

$$N = U\,(C_1'\, C_2')$$

The normal component of relative velocity between the two balls before collision:

$$V_{rel} = ((V_2 - V_1).N)\, N$$

Thus,

$V_1' = V_1 + \frac{m2}{\frac{m2+m1}{2}}\, V_{rel}$

$V_2' = V_2 - \frac{m1}{\frac{m2+m1}{2}}\, V_{rel}$

Comparison between PIC and PIT

Advantages of PIC:

- PIC is accurate as actual collision time is predicted in advance.
- It is not necessary to check for collision at every frame.

Limitations of PIC:

- It is not very easy to calculate the collision conditions when the bodies don't have uniform velocities or have rotations in addition to translations.
- It may be difficult to calculate the collision conditions when the bodies are irregularly shaped.

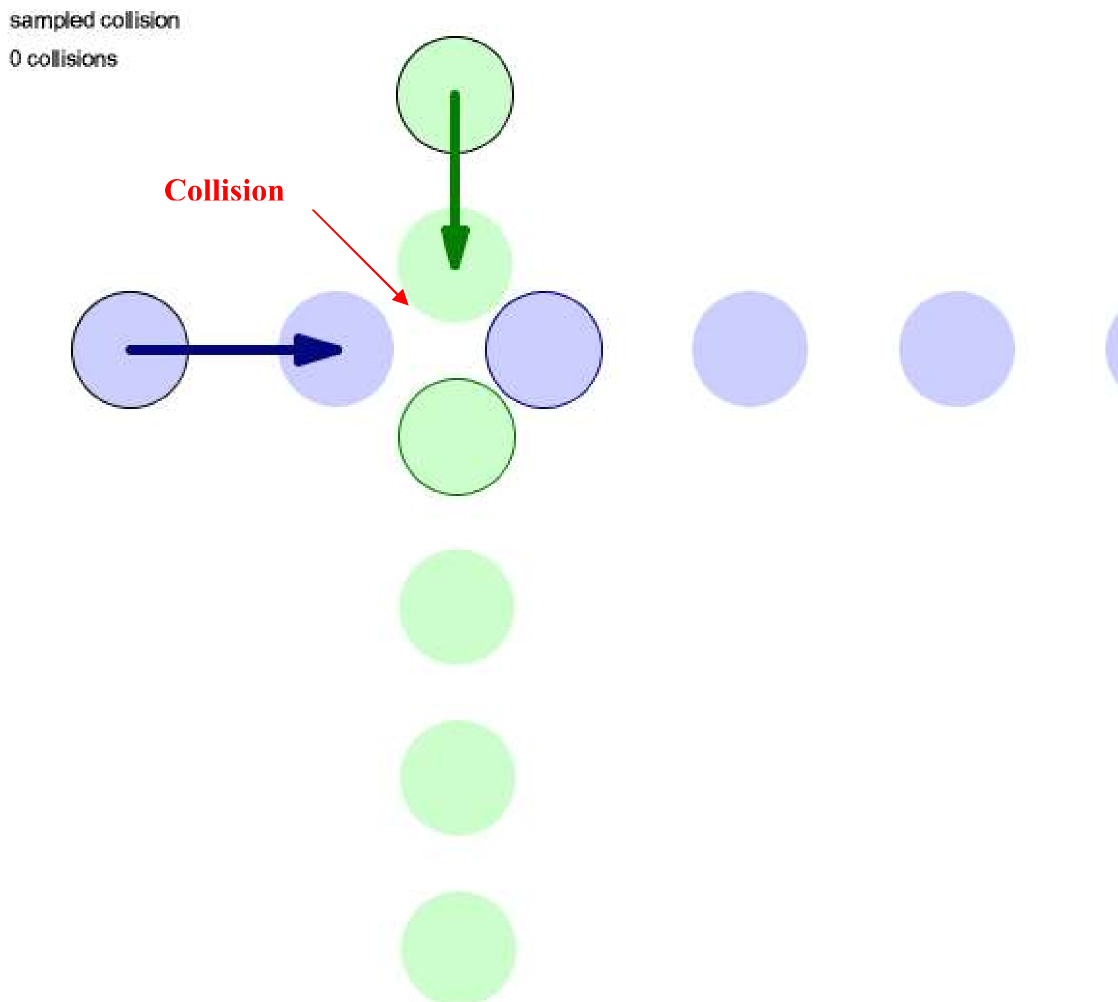
Advantages of PIT:

- PIT is not as computationally intensive as PIC. In PIT, we only need to check for interference where as in PIC collision equations need to be solved accurately.
- PIT is easily usable for bodies of any general shape.
- If the frame rate is high, PIT may be acceptable.

Limitations of PIT:

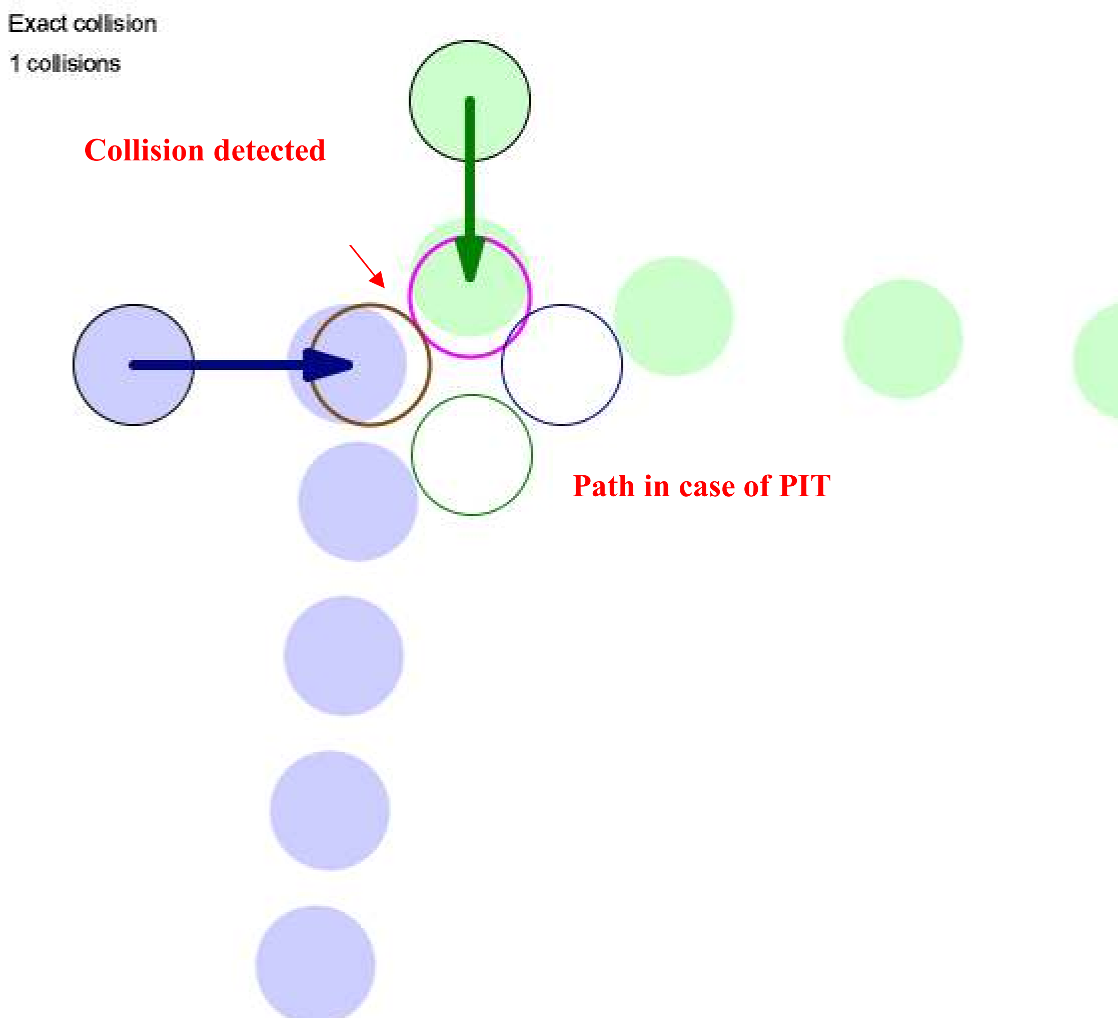
- Even when the frame rate in PIT is high, the results may not be accurate.
- The method is not very accurate. Particularly the following cases may arise, which are illustrated in Figures 2 thru 9.

Case 1: Collision Missed



(PIT)

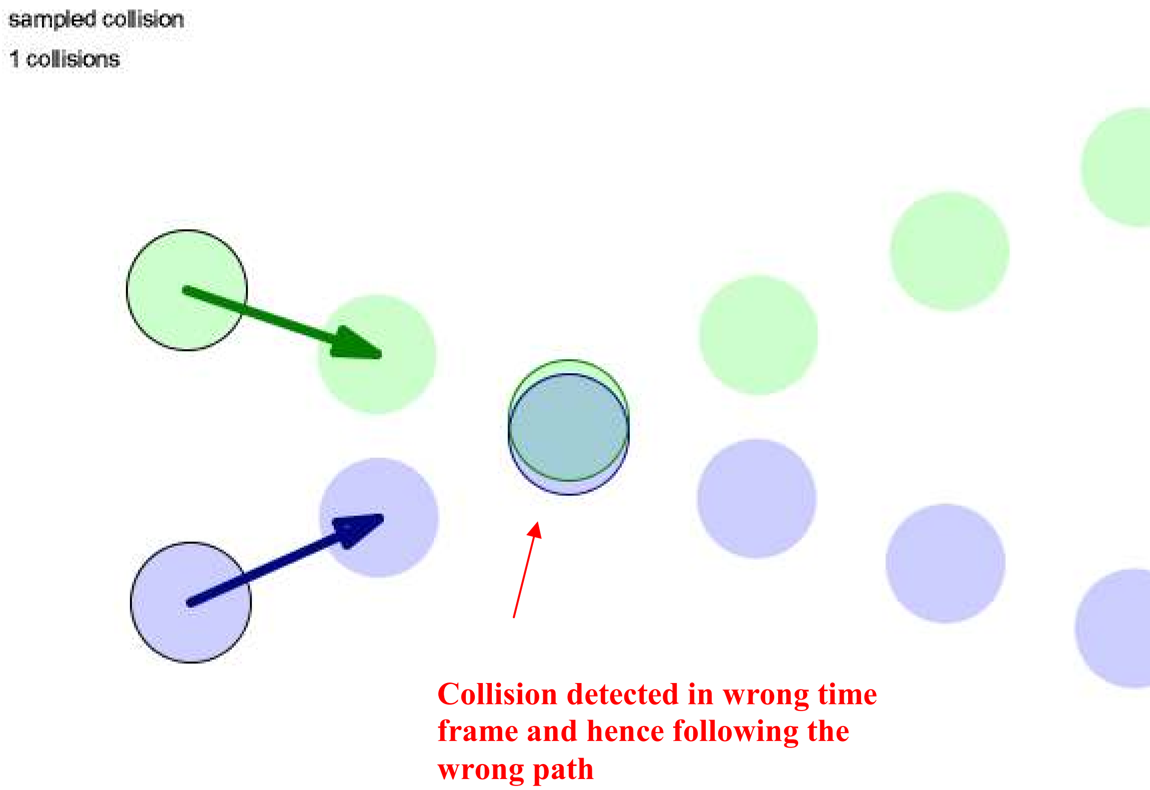
Figure 2: Collision is completely missed in case of PIT



(PIC)

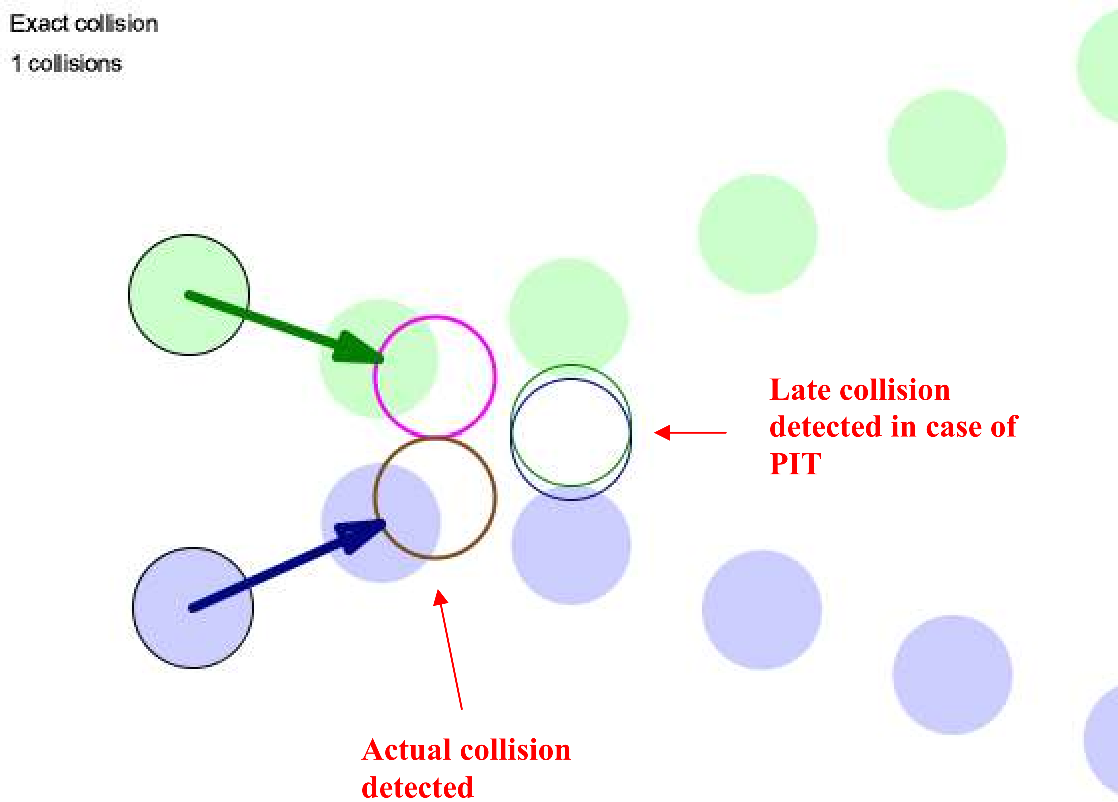
Figure 3: The collision is detected in PIC (Compare with Figure 2)

Case 2: Wrong time and path



(PIT)

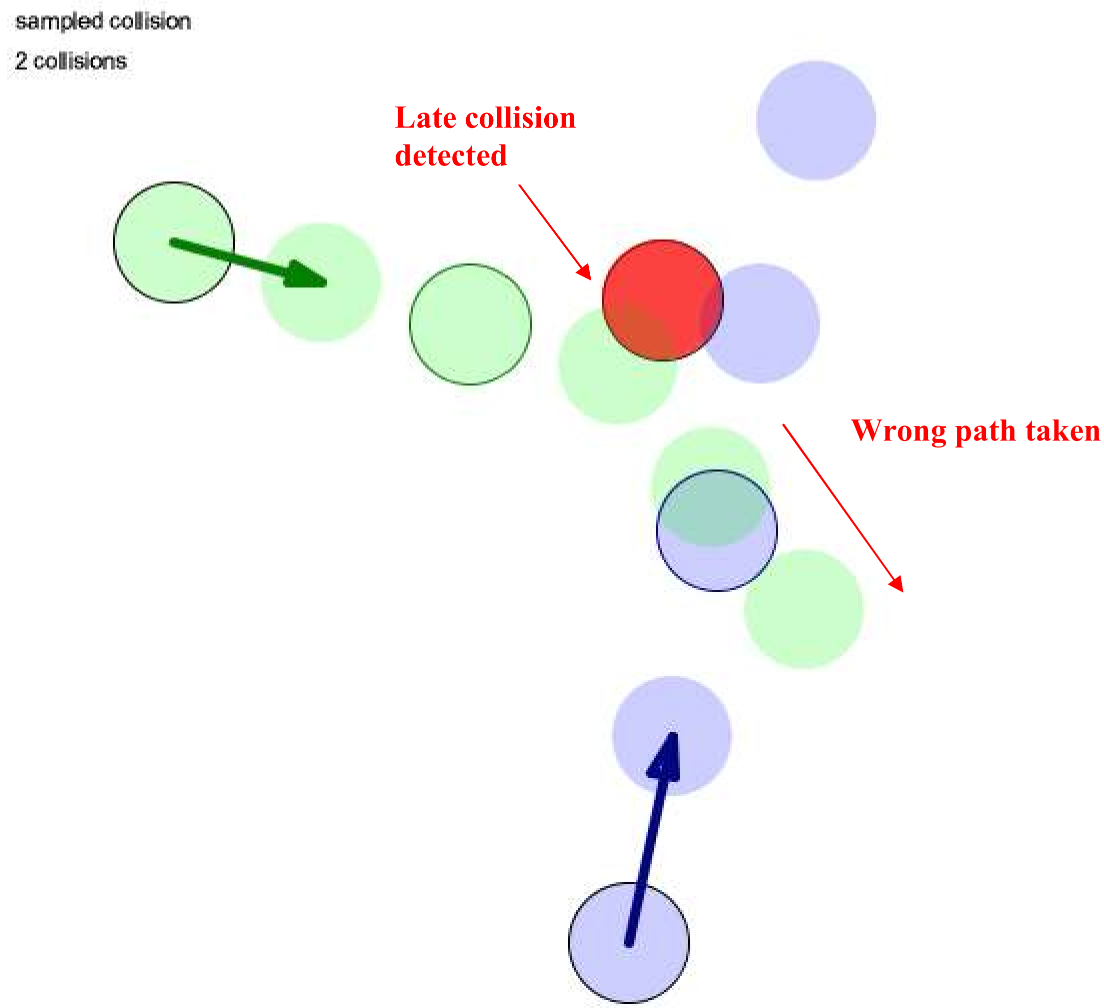
Figure 4: Collision detected too late in PIT. This causes the entire animation to appear unrealistic



(PIC)

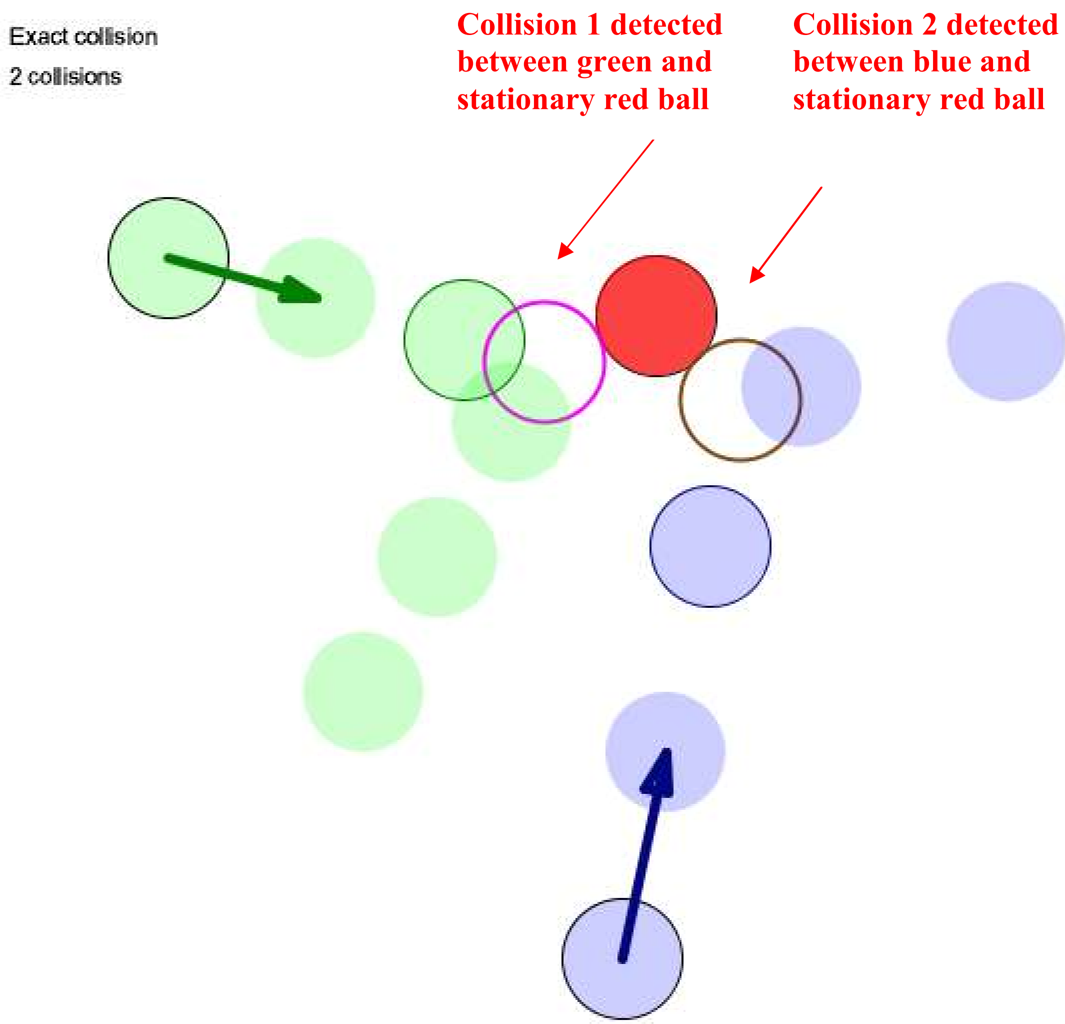
Figure 5: Exact time of collision calculated and implemented in PIC. Hence the animation appears real

Case 3: Wrong direction after bounce



(PIT)

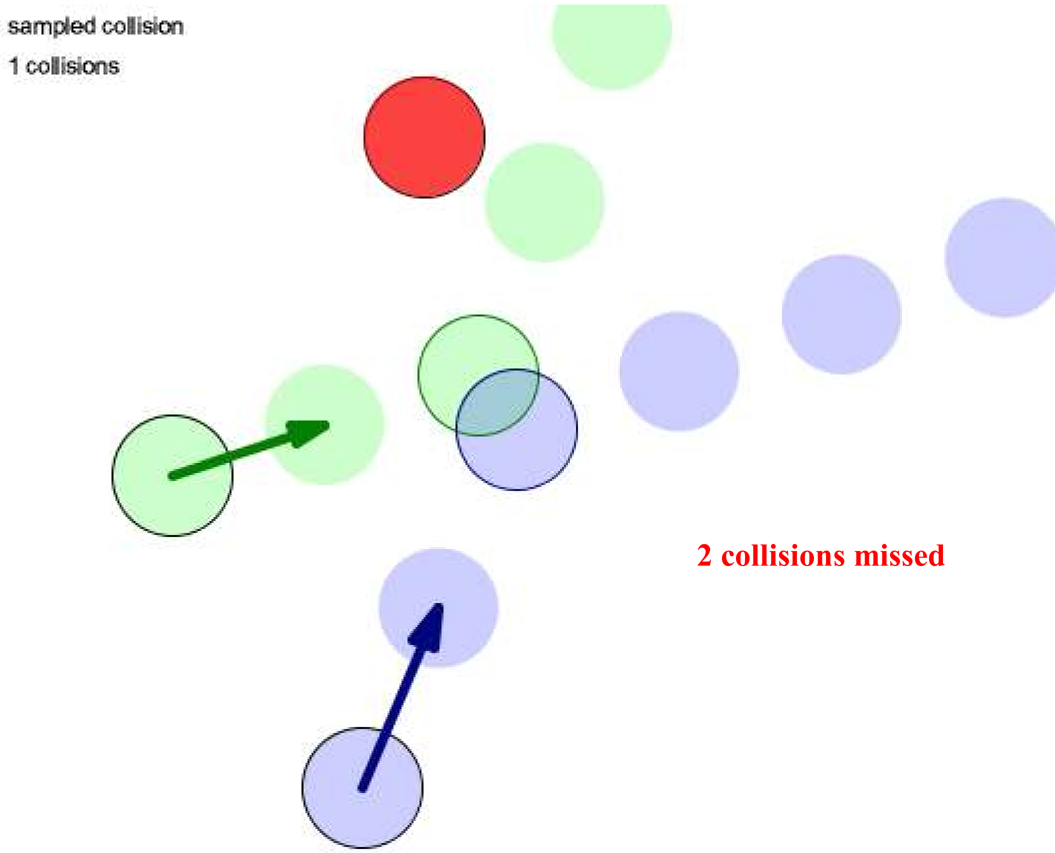
Figure 6: Direction after bounce is wrong in PIT because of miscalculation of the precise time of collision



(PIC)

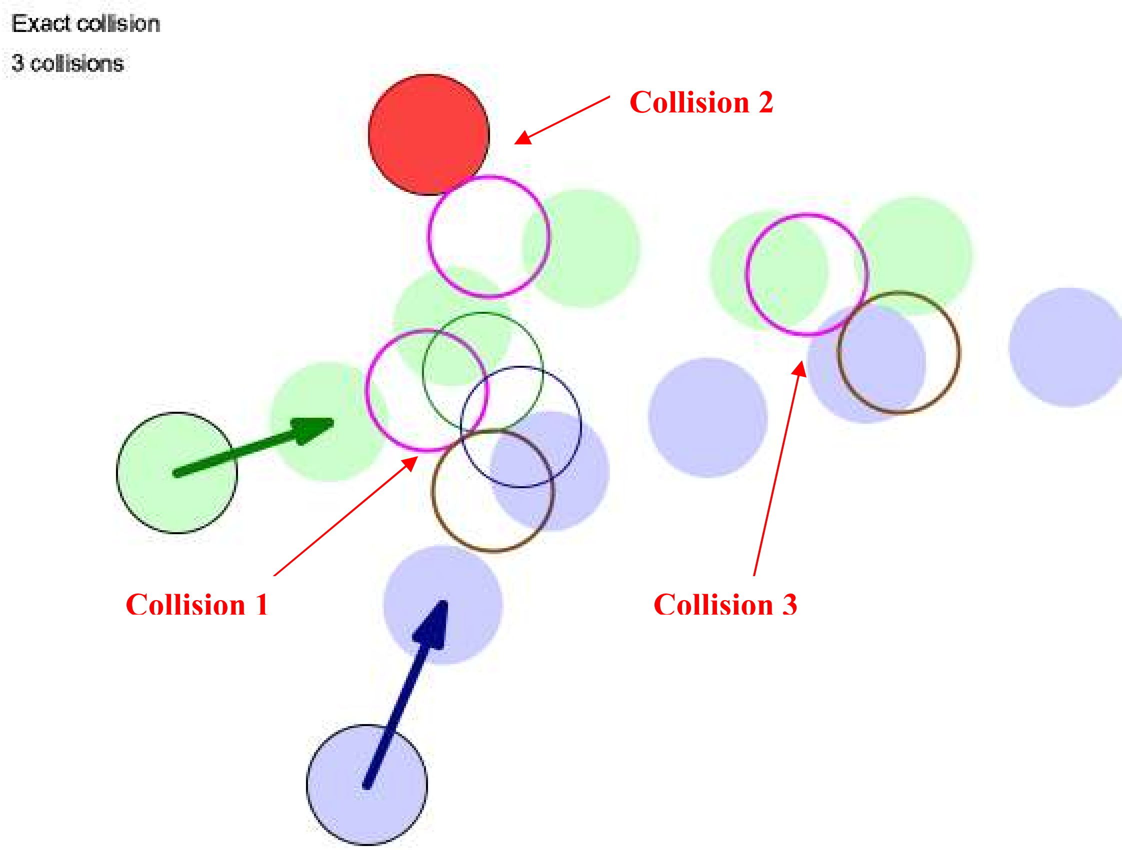
Figure 7: Direction of bounce correctly calculated due to precision of the time of collision in PIC

Case 4: Multiple Collisions



(PIT)

Figure 8: Multiple collisions are missed in case of PIT



(PIC)

Figure 9: Multiple collisions correctly detected in case of PIC

Conclusion

PIT may be acceptable when the frame rate is high. A high frame rate implies that the time instances are shorter. In such cases, detecting interference frequently may help reduce errors.

If bodies are not moving at constant velocity or are rotating or are not of regular shapes, the collision conditions are a lot more complicated to derive.