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Lecture 1: Introduction to Logic

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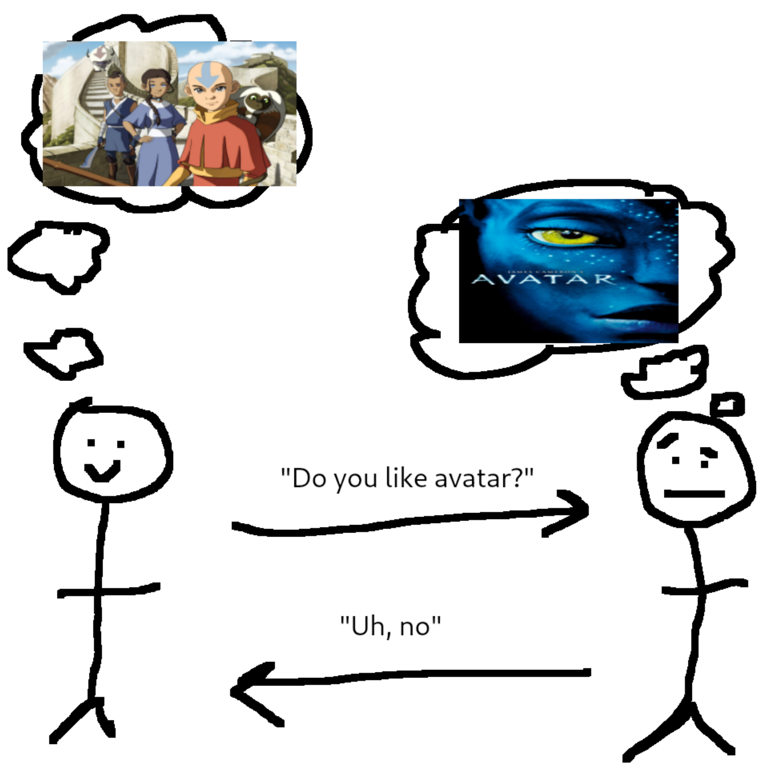
1 Why Logic?

Logic is a formalization of pure thought. Your brain has ideas, but in order to express those ideas, you must use language. It is also thought that the patterns and structure of language are also those of the patterns and structure of thought. Unfortunately, since language is a natural product, it contains some defects. Since the meaning of words cannot be fixed, it is not possible to exclude all possible misunderstandings. Consider the sentence "The dog ate the cat who scratched the fence. " We may say this sentence is ambiguous. There are many kinds of ambiguity.



Rather than expressing assertions, inferences, and deductions in natural language, we may express them in our formal language in order to limit ambiguity and fallacy. Mathematicians have created a "formal language", in that the meaning of these symbols cannot be misinterpretted. Deduction and reasoning can be performed in this formal language as one might perform arithmetic.

We will not extensively discuss why logic appears to emulate our own laws of thought, but rather the use of logic as a tool. As we define parts of this formal language, for each step, convince yourself like “yea, thats how I think”.

2 Truth

A proposition is a sentence which may be understood to have an objective truth value, one which can be demonstrated. The following are examples of propositions.

- Socrates is a man.
- The sun will rise tomorrow
- The sum of two numbers is a number.
- $10 > 7$
- $1 + 1 = 3$

These sentences may be understood to have an objective truth value. Some of them are false. The following sentences are not propositions:

- What time is it?
- Subscribe to me on youtube.
- Team X is the best.
- $n > 7$

Examples of sentences which are not propositions include questions, commands, and sentences with subjective truth value. For a truth to be objective, intuitively, you must understand that it is demonstrable. The third one cannot be assigned an objective truth value, as it is true or false for different values of n. This is a formula, and not a proposition. It can be made into a proposition and assigned a truth value by evaluation of n by a number.

Rather than pure english, we with to use a calculus of symbols so that the meaning of well formed formulas may be understood more objectively than natural language. To that extent, we use *propositional variables*. A propositional variable is a letter, such as p, q, r, s which represents a proposition. For example:

$$p := \text{Socrates is a man}$$

Now whenever p is referenced, it is in context of the proposition which asserts that Socrates is a man.

3 Negation

Note that intuitively, propositions may either be true or false. If a proposition is not true, it must be false. If a proposition is not false, it must be true. This is called *law of excluded middle*. For any proposition, it must be true, or its negation must be true. But what is the negation of a proposition?

The negation of a proposition p is written as $\neg p$. It is understood to be the logical opposite of proposition p. If p is truth, then the opposite of truth is certainly and can only be falsehood. So if p is true, then $\neg p$ is false, and if p is false, then $\neg p$ is true.

What is the opposite of white? If you said "black", thats incorrect. The opposite of "white" is "not white". For p the proposition that asserts "Socrates is a man". We may write $\neg p$ either as:

- It is not the case that Socrates is a man
- "Socrates is a man" is false.

Although these are correct representations, they are cumbersome in english. The proposition $\neg p$ may be equivalently rewritten as "Socrates is not a man". Note again that since p was true, $\neg p$ must be false.

What is the proposition which occurs when we negate a negated statement? Consider $\neg\neg p$. If p is true, then $\neg p$ is false, so then $\neg\neg p$ must be true. If p is true, then it is not the case that it is not the case that p is true, so we see that $\neg\neg p$ has identically the same truth value as p. The logical negations cancel each other out in a way similar to arithmetic!

4 Combining Propositions

You must observe that ideas may be formed from smaller, more atomic ideas.

4.1 Conjunction

Consider the proposition "Socrates is a man and all men are mortal". We use the english word "and" here to represent a combination of two smaller propositions. Let p be the proposition to assert that Socrates is a man, and q be the proposition to assert that all men are mortal. We may write the original proposition as the *conjunction* $p \wedge q$. How does the truth value of this conjunction depend upon its pieces? If you conjunct truths together, you can only get truth. If you conjunct several truths together, but then any falsehood, then the conjuncted proposition in whole must be false. For example, "p is true and p is false" is false. The proposition p must be either true or false, but it cannot be both. As another example, consider "Socrates is a man and I am Socrates". This proposition is false. Although Socrates is a man, I am not Socrates. As a whole, the statement is false, even if it contains some truth.

The word "and" is one english word we represent with conjunction, but others which are equivalent may include:

- p and q

- p but q
- p plus q
- p in addition to q

Natural language may distinguish between "and" and "but", but notice that logically, there is no difference with respect to the truth value of the established proposition.

4.2 Disjunction

Ideas may not need to be composed in such a way which requires all pieces, to be true, but perhaps just some, or any of the pieces. Consider the proposition "Socrates is a man or I am Socrates". Even though I am not socrates, the fact that Socrates is a man makes this proposition true on the whole. We may write the disjunction of propositions $p \vee q$ to represent a logical "or". A disjunction is true if any of its propositions are true. For example, the disjunction "p is true or p is false" is a true proposition, as p must definitely be either true or false. In a disjunction, if both are true, then the proposition is true.

Some other logically equivalent words to "or" include

- p otherwise q
- p rather q

4.3 Exclusive Or

Note the english use of the word "or" is not understood identically with the logical definition of "or". Sometimes, the use of "or" is exclusionary, in that "p or q" means "either p or q but not both". A logician goes to olive garden and the waiter asks, "would you like soup or salad with that?" The logician replies "yes". and is unfortunately charged extra for both.[1] This kind of "or" we denote as an exclusionary or, xor. We use the symbol $p \oplus q$ to mean xor.

5 Truth Tables

A truth table is a list where the first columns represent propositions and all possible truth values they may take on, and the later columns represent combinations of those propositions, and their respective truth values as well. Of the logical primitives we have seen so far, convince yourself that the following truth table is correct.

p	q	$p \vee q$	$p \wedge q$	$\neg p$	$\neg\neg p$	$p \oplus q$
T	T	T	T	F	T	F
T	F	T	F	F	T	T
F	T	T	F	T	F	T
F	F	F	F	T	F	F

if there are n distinct propositions, then a truth table will have 2^n rows. When filling out a truth table, please write all rows and columns.

[1]true story

5.1 Logical Consequence

Some propositions entail each other, in an if-then manner. Consider the proposition "If Socrates is a man, then all men are mortal". We may represent this as "If p then q", and use the notation $p \implies q$. This is called an implication and is read as "p implies q". We refer to p as the premise, or hypothesis. Then q is the conclusion, or consequence.

What is the truth value of an implication? Certainly if p is true, and q is false, then the implication $p \implies q$ should be false. But what about the other cases?

Consider the implication "If you study, you will pass"

- If p is true, and q is true, then $p \implies q$ should be true
- If p is true, and q is false, then $p \implies q$ should be false. If you study and fail, then the implication was not true.
- If p is false, and q is true, then $p \implies q$ is true. If you don't study and you pass, the implication is still true.
- If p is false, and q is false, then $p \implies q$ is true. If you don't study and you fail, the implication is still true.

We may represent this as the following truth table:

p	q	$p \vee q$	$p \wedge q$	$\neg p$	$\neg\neg p$	$p \oplus q$	$p \implies q$
T	T	T	T	F	T	F	T
T	F	T	F	F	T	T	F
F	T	T	F	T	F	T	T
F	F	F	F	T	F	F	T

There are many english equivalents

- p implies q
- If p then q
- A necessary condition for p is q
- A sufficient condition for p is q
- q given p
- q when p
- Ifp,q
- p only if q
- q follows from p

5.2 Related Implications

There are three related implications to $p \implies q$

- Given implication $p \implies q$, its converse is defined as $q \implies p$.
- The inverse of the implication $p \implies q$ is defined as $(\neg p) \implies (\neg q)$
- The contrapositive of the implication $p \implies q$ is defined to be $(\neg q) \implies (\neg p)$.

Lets add these to our truth table. Rather than compute these by understanding sentences, we may simply compute them using the previous entries in the truth table.

p	q	$p \vee q$	$p \wedge q$	$\neg p$	$\neg\neg p$	$p \oplus q$	$p \implies q$	$q \implies p$	$(\neg p) \implies (\neg q)$	$(\neg q) \implies (\neg p)$
T	T	T	T	F	T	F	T	T	T	T
T	F	T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	T	F	F	T
F	F	F	F	T	F	F	T	T	T	T

5.3 biconditionals

Notice the $p \implies q$ and its contrapositive have identical columns in the truth table. That implies that these two logical statements are equivalent. If two propositions are equivalent, we may write them as $p \iff q$ and say p if and only if q. This is called a biconditional, or a characterization. This is similar to the definition of equality in arithmetic. For example, "It rains if and only if it pours". We may write the equivalence of an implication with its contrapositive as

$$(p \implies q) \iff (\neg q \implies \neg p)$$

Also note that since $p \implies q$ and $q \implies p$ are necessary and sufficient, we can represent $p \iff q$ as

$$(p \implies q) \wedge (q \implies p)$$

In english, its equivalents are:

- p is necessary and sufficient for q
- p implies q and q implies p
- p iff[2] q

Note in english, it is not common for people to say "if and only if" when they mean that. The use of language in this case is imprecise.

[2]This is a mathematical short hand to mean "if and only if".

6 Precedence

In order for our formal language of propositional logic to remove ambiguity, we must define an ordering of the operators. Is $\neg p \implies q$ to mean $(\neg p) \implies q$ or $\neg(p \implies q)$? Much like PEMDAS for arithmetic, we have an equivalent for propositional calculus.

- parenthesis
- $\neg$
- $\wedge$
- $\vee$
- $\implies$
- $\iff$

7 Problems

Show $(\neg p \vee q) \iff (p \implies q)$