**CS 2050 Discrete Mathematics** March 12th 2024

# Lecture 18: More Induction and Strong Induction

*Lecturer: Abrahim Ladha*

Lets do two more examples of strong induction today

## 1 Fibonacci Numbers

We have shown previously you can use strong induction to prove closed formulas or recursively defined sequences. Today we prove a closed formula of one of the most popular recurrences, the Fibonacci numbers. Defined via the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

with two base cases $F_0 = 0, F_1 = 1$. You may surprised to know that it has a closed formula. The reason you've never been taught it is because, look how complicated the formula is:

**Theorem 1.**

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{1}$$

Before we prove the correctness of this formula, we will need a small lemma.

**Lemma 2.** For $\varphi_+ = \frac{1+\sqrt{5}}{2}$ and $\varphi_- = \frac{1-\sqrt{5}}{2}$ These two numbers satisfy $\varphi_+^2 = \varphi_+ + 1$ and $\varphi_-^2 = \varphi_- + 1$

*Proof.* Notice that $\varphi_+, \varphi_-$ are the two roots of $x^2 - x - 1 = 0$ by the quadratic formula, and thus satisfy $x^2 = x + 1$. □

*Proof.* Lets prove the base case of $n = 2$. By the currence, we get $F_2 = F_1 + F_0 = 1 + 0 = 1$, so we can prove the base case by plugging in $n = 2$, and confirming that we get 1.

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 =$$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2} + 1\right) =$$

$$\frac{1+\sqrt{5}}{2\sqrt{5}} + \frac{1}{\sqrt{5}} - \frac{1-\sqrt{5}}{2\sqrt{5}} - \frac{1}{\sqrt{5}} =$$

$$\frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} =$$

$$\frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

Now assume by strong induction that the formula is correct for $0, 1, 2, ..., k$. We prove that it is true for $k+1$. We prove that $F_{k+1} = \frac{1}{\sqrt{5}}\varphi_+^{k+1} - \frac{1}{\sqrt{5}}\varphi_-^{k+1}$

$$
\begin{aligned}
F_{k+1} = F_k + F_{k-1} &= \text{by strong induction} \\
\frac{1}{\sqrt{5}}\varphi_+^k - \frac{1}{\sqrt{5}}\varphi_-^k + \frac{1}{\sqrt{5}}\varphi_+^{k-1} - \frac{1}{\sqrt{5}}\varphi_-^{k-1} &= \\
\frac{1}{\sqrt{5}}\varphi_+^{k-1}(\varphi_+ + 1) - \frac{1}{\sqrt{5}}\varphi_-^{k-1}(\varphi_- + 1) &= \\
\frac{1}{\sqrt{5}}\varphi_+^{k-1}(\varphi_+^2) - \frac{1}{\sqrt{5}}\varphi_-^{k-1}(\varphi_-^2) &= \\
\frac{1}{\sqrt{5}}\varphi_+^{k+1} - \frac{1}{\sqrt{5}}\varphi_-^{k+1} &=
\end{aligned}
$$

as desired □

This is the power of proof by induction, proof in general. We have absolute certainty of the correctness of the formula. It doesn't matter if we know how to get the formula, we can prove that we know for certain it is right. This is a very weird formula, yet we know it is correct, even if we can't explain where it comes from!

## 2 Binary Trees

Lets do just one more simple example

**Definition 2.1.** A binary tree is full if every node has two or zero children

**Definition 2.2.** A binary tree is complete if all leaves are at the same level.

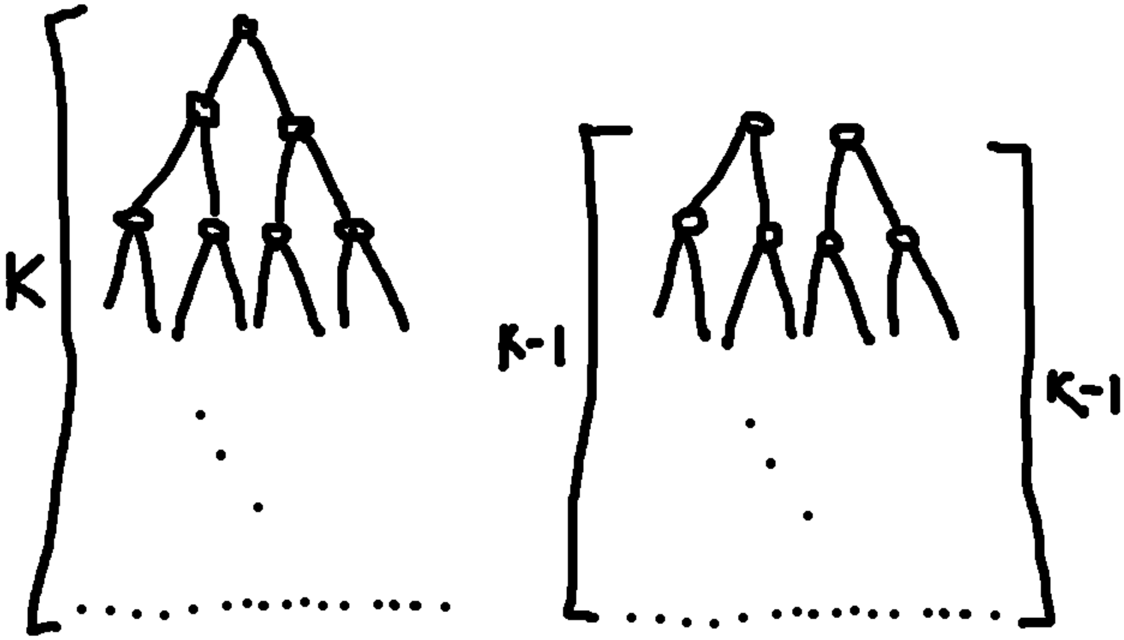
A binary tree is full and complete if it looks like the textbook picture of a generic binary tree, all leaves at the same, last level and no leaves are missing.

**Theorem 3.** A full complete binary tree of depth $k$ has $2^k$ leaves

Although this can be proved by a simple counting argument, for demonstration, we proof it by induction

*Proof.* We proceed by induction on the depth of the binary tree. Our base case is on binary trees of depth zero. A depth zero full and complete binary tree has one leaf, which is also the root. We see that $2^0 = 1$ and the base case is proved.

Now assume it is true for binary trees of depth $k-1$. We prove it is true for binary trees of depth $k$. Consider a full and complete binary tree of depth $k$. If you delete the root, you have two full and complete binary trees of depth $k-1$.




Notice that we haven't touched a leaf, so the number of leaves are the same. By the inductive hypothesis, These two binary trees of depth $k-1$ have $2^{k-1}$ leaves each, so the number of leaves of our depth $k$ tree was $2^{k-1} + 2^{k-1} = 2(2^{k-1}) = 2^k$. $\square$