CS 2050 Discrete Mathematics January 9th 2024

Lecture 2: Propositional Logic

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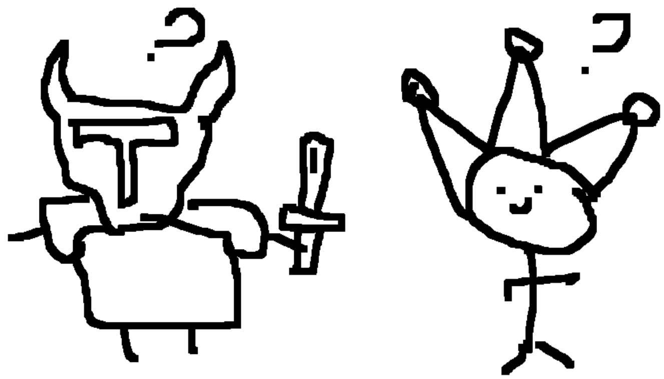
1 Logic Puzzles

If propositional logic is supposed to be a formalization of thought, then we should be able to apply it to solve some realistic problems. There are often small gotcha's, or confusing paradoxes. But many of these are not true paradoxes. Although they may seem contradictory, this vanishes when we can express the problem in the clear language that is the propositional calculus.

Consider the Knights and Knaves[1] problem.



You are in some sort of monty python skit, in which everyone is either a knight or a knave, and not both. Knights always tell the truth, and knaves always lie. You come across two travelers, we may denote as A, B.

- Person A says "B is a knight".
- Person B says "The two of us are opposite types"

If A is telling the truth, then B is a knight, so B is telling the truth, so then A must be a knave, and must be lying? See how on the surface it seems paradoxical? Lets try again by applying the propositional calculus.

Let p, q denote the propositions that "A is a knight" and "B is a knight" respectively. Then $\neg p, \neg q$ denote the propositions that "A is a knave" and "B is a knave" respectively. If you are not a knight, then you must be a knave.

We may represent the statements given then as

- Person A: $(p \implies q) \wedge (\neg p \implies \neg q)$
- Person B: $q \iff [(p \wedge \neg q) \vee (\neg p \wedge q)]$

If A is a knight, then p is true, so q must also be true, so p, q are both true. But q asserts that p, q must be different, so A cannot be a knight. If p is false, and B is a knave, then we know that B must also be a knave. Since B asserts that A, B must be different types, if they are lying then A, B must be the same type. So they actually were both jesters.

[1]The dictionary definition of a knave is "An unprincipled, crafty fellow."

We may determine the answer even more mechanically with a truth table. Observe that A statement can be simplified to $p \iff q$.

p	q	$\neg p$	$\neg q$	$p \iff q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$	$q \iff (p \wedge \neg q) \vee (\neg p \wedge q)$	
T	T	F	F	T	F	F	F	F	
T	F	F	T	F	T	F	T	F	
F	T	T	F	F	F	T	T	T	
F	F	T	T	T	F	F	F	T	

Lets look at the columns that are important, and take the conjunction of the statements given by both parties

p	q	$p \iff q$	$q \iff (p \wedge \neg q) \vee (\neg p \wedge q)$	$[p \iff q] \wedge [q \iff (p \wedge \neg q) \vee (\neg p \wedge q)]$	
T	T	T	F	F	
T	F	F	F	F	
F	T	F	T	F	
F	F	T	T	**T**	

There is only one valid solution, both p, q are false, so they must both be knaves. Note that if there if the last column has no solution, then there is no solution to the puzzle. If the last column has more than one truth value, then there may be more than one solution to the problem.

truth table equivalence example

2 Equivalence

We may use the symbol $\equiv$ to denote that two propositions are equivalent, in a truth table sense. For example, we will show that

$$p \implies q \equiv \neg p \vee q$$

p	q	$\neg p$	$\neg p \vee q$	$p \implies q$	
T	T	F	**T**	**T**	
T	F	F	**F**	**F**	
F	T	T	**T**	**T**	
F	F	T	**T**	**T**	

Notice that the two last bolded columns are the same. Their respective propositions are said to be equivalent.

3 Laws of Thought

If two propositions are equivalent, then you may replace one for the other in some larger proposition. We can modify the proposition into a smaller one this way, and determine its truth with a smaller truth table. We will demonstrate the equivalence of all our rules using truth table, but afterwards, you may take them like you take the laws of arithmetic. You use $a + b = b + a$ for example without prejudice. Some of these laws we can show by a truth table equivalence, others are so simple, we can only take them as laws.

3.1 Identity

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

3.2 Domination

- $p \lor T \equiv T$
- $p \land F \equiv F$

3.3 Idempotent

- $p \land p \equiv p$
- $p \lor p \equiv p$

3.4 Double Negation

- $\neg\neg p \equiv p$

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

3.5 Communitivity

- $p \land q \equiv q \land p$
- $p \lor q \equiv q \lor p$

3.6 Associativity

- $(p \land q) \land r \equiv p \land (q \land r)$
- $(p \lor q) \lor r \equiv p \lor (q \lor r)$

p	q	r	$(p \land q)$	$(q \land r)$	$(p \land q) \land r$	$p \land (q \land r)$	$(p \lor q)$	$(q \lor r)$	$(p \lor q) \lor r$	$p \lor (q \lor r)$
T	T	T	T	T	**T**	**T**	T	T	**T**	**T**
T	T	F	T	F	**F**	**F**	T	T	**T**	**T**
T	F	T	F	F	**F**	**F**	T	T	**T**	**T**
T	F	F	F	F	**F**	**F**	T	F	**T**	**T**
F	T	T	F	T	**F**	**F**	T	T	**T**	**T**
F	T	F	F	F	**F**	**F**	T	T	**T**	**T**
F	F	T	F	F	**F**	**F**	F	T	**T**	**T**
F	F	F	F	F	**F**	**F**	F	F	**F**	**F**

3.7 Distributive Laws

- $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
- $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$(p \wedge q) \vee (p \wedge r)$	$(q \vee r)$	$p \wedge (q \vee r)$
T	T	T	T	T	**T**	T	**T**
T	T	F	T	F	**T**	T	**T**
T	F	T	F	T	**T**	T	**T**
T	F	F	F	F	**F**	F	**F**
F	T	T	F	F	**F**	T	**F**
F	T	F	F	F	**F**	T	**F**
F	F	T	F	F	**F**	T	**F**
F	F	F	F	F	**F**	F	**F**

p	q	r	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$	$(q \wedge r)$	$p \vee (q \wedge r)$
T	T	T	T	T	**T**	T	**T**
T	T	F	T	T	**T**	F	**T**
T	F	T	T	T	**T**	F	**T**
T	F	F	T	T	**T**	F	**T**
F	T	T	T	T	**T**	T	**T**
F	T	F	T	F	**F**	F	**F**
F	F	T	F	T	**F**	F	**F**
F	F	F	F	F	**F**	F	**F**

3.8 DeMorgan's Laws

- $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
- $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
T	T	F	F	T	**F**	**F**	T	**F**	**F**
T	F	F	T	F	**T**	**T**	T	**F**	**F**
F	T	T	F	F	**T**	**T**	T	**F**	**F**
F	F	T	T	F	**T**	**T**	F	**T**	**T**

3.9 Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

p	q	$p \wedge q$	$p \vee q$	$p \vee (p \wedge q)$	$p \wedge (p \vee q)$
T	T	T	T	**T**	**T**
T	F	F	T	**T**	**T**
F	T	F	T	**F**	**F**
F	F	F	F	**F**	**F**

3.10 Implications

- $p \implies q \equiv \neg p \vee q$ (conditional disjunction equivalence)
- $p \implies q \equiv \neg q \implies \neg p$ (See contrapositive from previous lecture)
- $p \vee q \equiv \neg p \implies q$
- $p \wedge q \equiv \neg(p \implies \neg q)$

- $\neg(p \implies q) \equiv p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \implies q$	$p \implies \neg q$	$\neg(p \implies \neg q)$	$p \wedge q$	$p \implies q$	$\neg(p \implies q)$	$p \wedge \neg q$
T	T	F	F	**T**	**T**	F	**T**	**T**	T	**F**	**F**
T	F	F	T	**T**	**T**	T	**F**	**F**	F	**T**	**T**
F	T	T	F	**T**	**T**	T	**F**	**F**	T	**F**	**F**
F	F	T	T	**F**	**F**	T	**F**	**F**	T	**F**	**F**

- $(p \implies q) \wedge (p \implies r) \equiv p \implies (q \wedge r)$
- $(p \implies r) \wedge (q \implies r) \equiv (p \vee q) \implies r$
- $(p \implies q) \vee (p \implies r) \equiv p \implies (q \vee r)$
- $(p \implies r) \vee (q \implies r) \equiv (p \wedge q) \implies r$

p	q	r	$(p \implies q)$	$(p \implies r)$	$(q \wedge r)$	$p \implies (q \wedge r)$	$(p \implies q) \wedge (p \implies r)$	...
T	T	T	T	T	T	**T**	**T**	
T	T	F	T	F	F	**F**	**F**	
T	F	T	F	T	F	**F**	**F**	
T	F	F	F	F	F	**F**	**F**	
F	T	T	T	T	T	**T**	**T**	
F	T	F	T	T	F	**T**	**T**	
F	F	T	T	T	F	**T**	**T**	
F	F	F	T	T	F	**T**	**T**	

$(q \implies r)$	$p \vee q$	$(p \implies r) \wedge (q \implies r)$	$(p \vee q) \implies r$	...
T	T	**T**	**T**	
F	T	**F**	**F**	
T	T	**T**	**T**	
T	T	**F**	**F**	
T	T	**T**	**T**	
F	T	**F**	**F**	
T	F	**T**	**T**	
T	F	**T**	**T**	

$(p \implies q) \vee (p \implies r)$	$p \implies (q \vee r)$	$(p \implies r) \vee (q \implies r)$	$(p \wedge q) \implies r$
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**T**	**T**	**T**
F	**F**	**T**	**T**
T	**T**	**T**	**T**
T	**T**	**T**	**T**
T	**T**	**T**	**T**
T	**T**	**T**	**T**

3.11 Biconditionals

- $p \iff q \equiv (p \implies q) \wedge (q \implies p)$
- $p \iff q \equiv \neg p \iff \neg q$

- $p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \iff q) \equiv (p \iff \neg q)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$q \implies p$	$p \wedge q$	$\neg p \wedge \neg q$	...
T	T	F	F	T	T	T	F	
T	F	F	T	F	T	F	F	
F	T	T	F	T	F	F	F	
F	F	T	T	T	T	F	T	

$p \iff q$	$\neg p \iff \neg q$	$(p \implies q) \wedge (q \implies p)$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$\neg(p \iff q)$	$(p \iff \neg q)$
T	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**	**T**
T	**T**	**T**	**T**	**F**	**F**

4 Tautologies and Contradictions

A tautology is a proposition which is always true. A contradiction is a proposition which is always false. $p \vee \neg p$ is a canonical example of a tautology and $p \wedge \neg p$ is a canonical example of a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

5 Examples

So far, we know we can demonstrate two propositions to be equivalent by computing their truth tables and observing they have the same columns. But we don't need to do this, we can do without this by simply applying our previously demonstrated laws.

Suppose we want to demonstrate that $\neg(p \implies q)$ is equivalent to $p \wedge \neg q$. We can do this without a truth table as follows

$$\neg(p \implies q) \equiv \quad \text{conditional disjunction equivalence} \quad (1)$$
$$\neg(\neg p \vee q) \equiv \quad \text{DeMorgan's} \quad (2)$$
$$\neg(\neg p) \wedge \neg q \equiv \quad \text{Double Negation} \quad (3)$$
$$p \wedge \neg q \equiv \quad (4)$$

Suppose we want to demonstrate that $(p \wedge q) \implies (p \vee q)$ is always true, it is a tautology.

$$(p \wedge q) \implies (p \vee q) \equiv \quad \text{conditional disjunction equivalence} \quad (5)$$
$$\neg(p \wedge q) \vee (p \vee q) \equiv \quad \text{DeMorgan's} \quad (6)$$
$$(\neg p \vee \neg q) \vee (p \vee q) \equiv \quad \text{Associativity} \quad (7)$$
$$(\neg p \vee p) \vee (\neg q \vee q) \equiv \quad \text{Negation} \quad (8)$$
$$T \vee T \equiv \quad (9)$$
$$T \quad (10)$$

Rather than doing very complicated and tedious truth tables, we can demonstrate equivalence by applying laws. We demonstrate the absorption law $p \vee (p \wedge q) \equiv p$

$p \vee (p \wedge q) \equiv$	Identity	(11)
$(p \wedge T) \vee (p \wedge q) \equiv$	Distributive	(12)
$p \wedge (T \vee q) \equiv$	Domination	(13)
$p \wedge T \equiv$	Identity	(14)
p		(15)
		(16)

6 All Laws

Here is a small cheatsheet of exactly and only the laws you can use.

Identity

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

Domination

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

Idempotent

- $p \wedge p \equiv p$
- $p \vee p \equiv p$

Double Negation

- $\neg\neg p \equiv p$

Communitivity

- $p \wedge q \equiv q \wedge p$
- $p \vee q \equiv q \vee p$

Associativity

- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

DeMorgan's Laws

- $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
- $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

Implication

- $p \implies q \equiv \neg q \implies \neg p$ — contrapositive
- $p \implies q \equiv \neg p \vee q$ — conditional disjunction equivalence

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Biconditional

- $p \iff q \equiv (p \implies q) \wedge (q \implies p)$

7 Other Laws

These are some laws which may be demonstrated from those previous. You may not apply these, but you should know them.

- $p \lor q \equiv \neg p \implies q$
- $p \land q \equiv \neg(p \implies \neg q)$
- $\neg(p \implies q) \equiv p \land \neg q$
- $(p \implies q) \land (p \implies r) \equiv p \implies (q \land r)$
- $(p \implies r) \land (q \implies r) \equiv (p \lor q) \implies r$
- $(p \implies q) \lor (p \implies r) \equiv p \implies (q \lor r)$
- $(p \implies q) \lor (p \implies r) \equiv (p \land q) \implies r$
- $p \iff q \equiv (p \implies q) \land (q \implies p)$
- $p \iff q \equiv \neg p \iff \neg q$
- $p \iff q \equiv (p \land q) \lor (\neg p \land \neg q)$
- $\neg(p \iff q) \equiv (p \iff \neg q)$