CS 2050 Discrete Mathematics April 4th 2024

Lecture 22: More Pigeonhole

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1 Pigeonhole Principle

Recall the pigeonhole principle

Definition 1.1 (Pigeonhole Principle)**.** If you have n pigeons and k pigeonholes, then for any assignment of the pigeons into the holes, there exists a hole with > 1 pigeons assigned to it.

The generalized pigeonhole principle is an obvious generalization on size, which we proved last time.

Definition 1.2 (Generalized Pigeonhole)**.** If you have n pigeons and k holes, then there exists a hole with $\lceil \frac{n}{k} \rceil$ pigeons.

2 Examples

2.1 Program

Suppose you have a computer program to randomly spit out natural numbers. What is the fewest numbers must it spit out so that you may guarantee three of them end in the same digit?

If you take a number and mod it by 10, it falls into one of the 10 equivalence classes $0, ..., 9$. We are looking for some n such that $\lceil \frac{n}{10} \rceil \geq 3$ but $\lceil \frac{n-1}{10} \rceil \leq 2$. Then we see that $n = 21$.

One way to think about this is if you were to put 2 numbers into 10 holes, the 21st number is forced in some hole that already has 2 numbers.

2.2 File Compression

Consider a lossless file compression algorithm such as the zip file algorithm. A lossless file compression algorithm can be thought of as a bijection such that when it compresses a file x, it outputs another file $x.zip$. It can decompress as well to get back the original file. Functionally, we may say $f(x) = x.zip$ and $f^{-1}(x.zip) = x$. It is useful to output smaller files to save space. We may say a file is compressible if $|x.zip| < |x|$. We prove that not all files are compressible.

Theorem 1. For every lossless file compression algorithm and for every length n and there exists a file of n bits such that $|x.zip| \geq |x|$.

Proof. We proceed by a counting argument. Let n be any number, and consider the number of files of n bits. This is exactly the number of bit strings of length n, so 2^n. How many possible outputs are there of $x.zip$ if $|x.zip| < |x| = n$? We see that $x.zip$ may take on lengths $0, 1, ..., n-1$ if it is smaller than x. How many files of length $\leq n-1$ are there?

$$\text{possible x.zip outputs} \leq \sum_{i=0}^{n-1} \text{files of length } i = \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

If $|x.zip| < n$ then there are $2^n - 1$ possible output files. But there are 2^n possible inputs. By the pigeonhole principle, some input file is not mapped to a smaller output. so there exists some string x of length n such that $|x.zip| \geq |x|$ □

Again, pigeonhole did not tell us which file is incompressible, how many of them are incompressible, it was only able to tell us that such files must exist. The pigeonhole principle has willed incompressible files into existence.

2.3 Square

Theorem 2. Suppose you have a square $[0, 2] \times [0, 2]$. If you place five points in the square, then a pair of them will be no more than $\leq \sqrt{2}$ apart.

Proof. Consider cutting the square into four $[0, 1] \times [0, 1]$ squares. There are four squares and five points. By the pigeonhole principle, there exists some square with two points. Within this square, the farthest distance they could be is at opposite corners. Their distance apart is then at most the length of the hypotenuse of a square of side lengths 1. By the pythagorean theorem, there distance apart is then $\leq \sqrt{1^2 + 1^2} = \sqrt{2}$. □

Theorem 3. If you place 101 points in a 6×8 rectangle, then two points must be within ≤ 1 of each other.

Proof. Cut the 6×8 rectangle into 100 0.6×0.8 rectangles. Since there are 101 points and 100 rectangles, by the pigeonhole principle, some rectangle must contain two points. The farthest these two points could be apart is the hypothenuse, which has length $\leq 0.6^2 + 0.8^2 = 1$. □

2.4 Coloring of the plane

Consider the real cartesian plane. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$.

Theorem 4. If you color every point of the cartesian plane either red or blue, then for every distance $d > 0$, there exists two points of the same color d apart.

Proof. Color every point of $\mathbb{R}^2$. Consider an equilateral triangle of side lengths d. Note that every point of our triangle is exactly d from the other two points. Put this triangle anywhere in the plane. Color the three points according to the coloring of the plane, according to the position they reside. Since there are three points and two colors, by the pigeonhole principle, two points must be the same color, and we see these two points are d apart. □

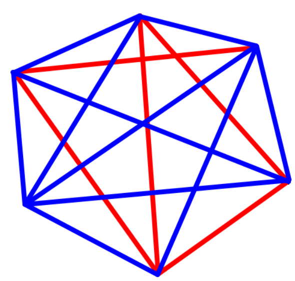
Again, the power of the pigeonhole principle! How many possible colorings are there of the two dimensional plane? How many possible strategies are there to color? No matter how you do it, the pigeonhole enforces that

Theorem 5. Consider an 8×8 chessboard, but you remove two opposite corners. There is no way to cover this board with dominoes

A domino is a 2×1 tile piece, which covers exactly two adjacent (non-diagonal) squares. A covering of a board with dominoes is a way to place dominoes over the board such that no square of the board is uncovered and also no dominoes are overlapping.

Proof. If you remove two opposing corners from a square chessboard, note that they must be the same color. Our board then has exactly 32 of one color and 30 of the other color. Without loss of generality, suppose there are 32 black squares and 30 white squares. Because of the checkered pattern, a domino must cover exactly two squares, and they must be of opposite color. Assume to the contrary our 62 square board can be covered by dominoes. Then this covering must use exactly 31 dominoes. Since there are 32 black squares and 31 dominoes, by the pigeonhole principle, some domino must cover two black squares. But this is impossible, contradiction. □

Consider the graph K_6. It has 6 vertices, and between each pairs of verticies is an edge, for a combined 15 edges.

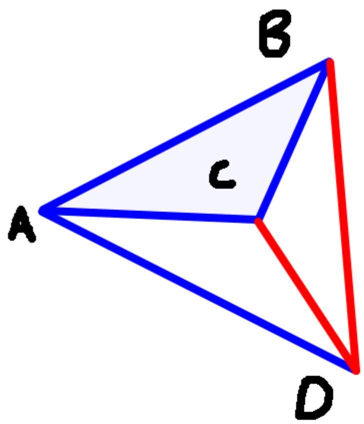


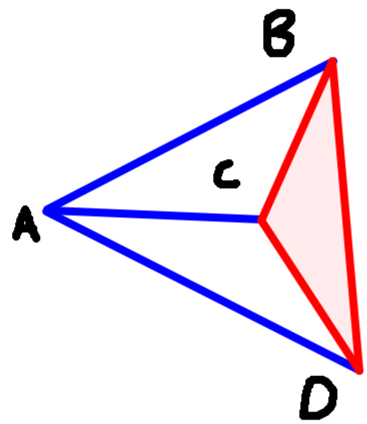
Theorem 6. If you color all the edges of K_6 either red or blue, then it must contain either a red triangle or a blue triangle.

Proof. Consider any edge coloring of K_6 with red and blue edges. Label the vertices A, B, C, D, E, F. Consider the 5 edges connected to A. Since there are five edges, and 2 colors, by the generalized pigeonhole principle, $\lceil \frac{5}{2} \rceil = 3$ of these edges are the same color. Without loss of generality, suppose they are blue, and connected to B, C, D. There are three edges between B, C, D themselves. We have two cases:

- If any of the edges between B, C, D are blue, then this edge forms a blue triangle with two of the three edges connected to A.
- If none of the edges between B, C, D are blue, then they are all red. Then this is a red triangle.

□





3 Further Advice

The pigeonhole principle is an exact statement, and can apply to a huge diversity of situations. It simply asserts that something exists. You don't know where it is or how much it has or anything useful. Simply by certain structures being what they are, the conditions force into existence something. They are willed. It is simultaneously a very powerful tool, but also not powerful enough.