CS 2050 Discrete Mathematics January 16th 2024

Lecture 3: Predicates and Quantifiers

Lecturer: Abrahim Ladha

1 Predicate

Recall that we had the example of a declarative sentence "$n > 7$" and that it was not a proposition. Its truth value relied on a "free variable" called n. Were we to fix this variable, then the truth value could be determined, and it would be a proposition. This is called a *predicate*, or a *propositional function*.

We may write this as

$$P(n) : n > 7$$

and use capital letters for predicates. Notice that $P(10) \equiv T$, but $P(2) \equiv F$. A predicate may have multiple arguments, such as $P(x, y, z) : x + y = z$. For a predicate to become a proposition, it must be the case it has no free variables. $P(1, 2, z)$ is a predicate, but not a proposition. $P(1, 2, 3)$ is a proposition, and is true.

The *universe of discourse*[1] must be defined for the variables of predicates. There is an understood set of possible values each free variable of a predicate can take on. Without this, the predicate is simply undefined.

Consider the predicate

$$P(x, y) : \text{If } x > 0 \text{ then } x + y = 10$$

. Observe that

- $P(-1, 100)$ is true
- $P(4, 6)$ is true
- $P(4, 5)$ is false

2 Quantification

Some words used in declarative english sentences include "Any, all, some, none, few," and so on. A quantifier specifies how a variable of a predicate interacts logically with the universe of discourse the variable ranges over.

3 Existential Quantification

The existential quantifier corresponds to the english words "there exists, some, atleast one" and so on. Consider the predicate

[1] Also called the universe, the domain of discourse, or the domain

$$P(x, y) : \text{elephant x is heavier than duck y}$$

The universe of discourse of x is all possible elephants, and the universe of discourse of y is all possible ducks. What is a predicate for the sentence 'Some elephant is heavier than duck y"? This is a predicate, but we determine that the elephant must exist, so its truth is dependent only upon y. We may express this using the symbol "$\exists$" and write three examples.

- $\exists x P(x, y)$: There exists an elephant which is heavier than duck y
- $\exists y P(x, y)$: elephant x is heavier than some duck
- $\exists x \exists y P(x, y)$: Some elephant is heavier than some duck.[2]

Note that the third example not a predicate, but is a proposition! The quantifier $\exists$ *binds* to the variable, making it no longer free. Once a predicate has all its variables bound, it is a proposition. We can demonstrate the truth of this proposition by finding just one elephant and just one duck such that the elephant weighs more than the duck. Again, the universe of discourse must be defined for the quantifier to make sense. In our previous example, the universe of x is all possible elephants, and the universe of y is all possible ducks.

Consider the proposition $\exists x[x \text{ is even}]$. This is true even if $P(x)$ is false for some values of x. Since we know there is atleast one even number, then we know this is true. There is no claim as to which numbers are even, or how to find them, simply that an even number exists.

3.1 Uniqueness

In language, we often want to denote not only that an item exists, but does so uniquely. To this extent, we use the quantifier $\exists!$ to denote this. For example "exactly one number x is positive", we may represent as $\exists! P(x)$. This is not a real quantifier, but you should know the notation if you come across it.

4 Universal Quantification

While the existential quantifier logically captures the meaning of words like "there is, atleast one, some, there exists" and so on, what about words like "every, for all, for each"? For these, we use the universal quantifier. Let $P(x)$ be a predicate. We write

$$\forall x P(x)$$

to mean that for every single possible value that x could take on from its universe of discourse, the predicate $P(x)$ is true. For example consider the sentence "every elephant is heavier than duck y". We may write this as

$$\forall x P(x, y) : \text{Every elephant is heavier than duck y}$$

[2]Most formally, this would be read as "There exists an elephant and there exists a duck such that the elephant is heavier than the duck

We may also quantify over the variable y to get

$$\forall x \forall y P(x,y) : \text{Every elephant is heavier than every duck}$$

.

Consider the Predicate $(x^2 \geq 0)$. We can bind its free variable to get the proposition $\forall x(x^2 \geq 0)$. Note how important the universe of discourse is. If the universe is a restriction of real numbers, then its true. If the universe involves complex numbers, its false.

5 Equivalence

We may say two statements[3] are equivalent if and only if they have the same truth values, regardless of propositions used, or universes of discourse allowed for the variables of the predicates. We denote $S \equiv T$ to mean these two statements are equivalent.

Let $P(x), Q(x)$ be two predicates, with the same variable over some universe of discourse. We demonstrate

$$\forall x(P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$$

Suppose that $\forall x(P(x) \wedge Q(x))$ is true. Then for all a in the universe, we know that $P(a) \wedge Q(a)$ is true. So both $P(a)$, $Q(a)$ are true. Since $P(a)$ is true and $Q(a)$ is true for any a in the univese, we know that $\forall x P(x)$ is true, and $\forall x Q(x)$ is true. So $(\forall x P(x)) \wedge (\forall x Q(x))$ is true.

Suppose that $(\forall x P(x)) \wedge (\forall x Q(x))$ is true. Then $(\forall x P(x))$ is true, and $(\forall x Q(x))$ is true. Then since they share the universe, we know that for every a that $P(a)$ is true and $Q(a)$ is true. So $P(a) \wedge Q(a)$ is true. Since a is any element in the universe of x, we see that $\forall x(P(x) \wedge Q(x))$ is true.

We can write the uniqueness quantifier equivalently as just an existential one

$$\exists ! P(x) \equiv \exists x[P(x) \wedge \forall y(y \neq x \implies \neg P(y))]$$

To interpret this back in natural language, it states that $\exists x P(x)$, for every other distinct value y, that $\neg P(x)$. This is the definition of uniqueness.

6 Multiple Quantifiers

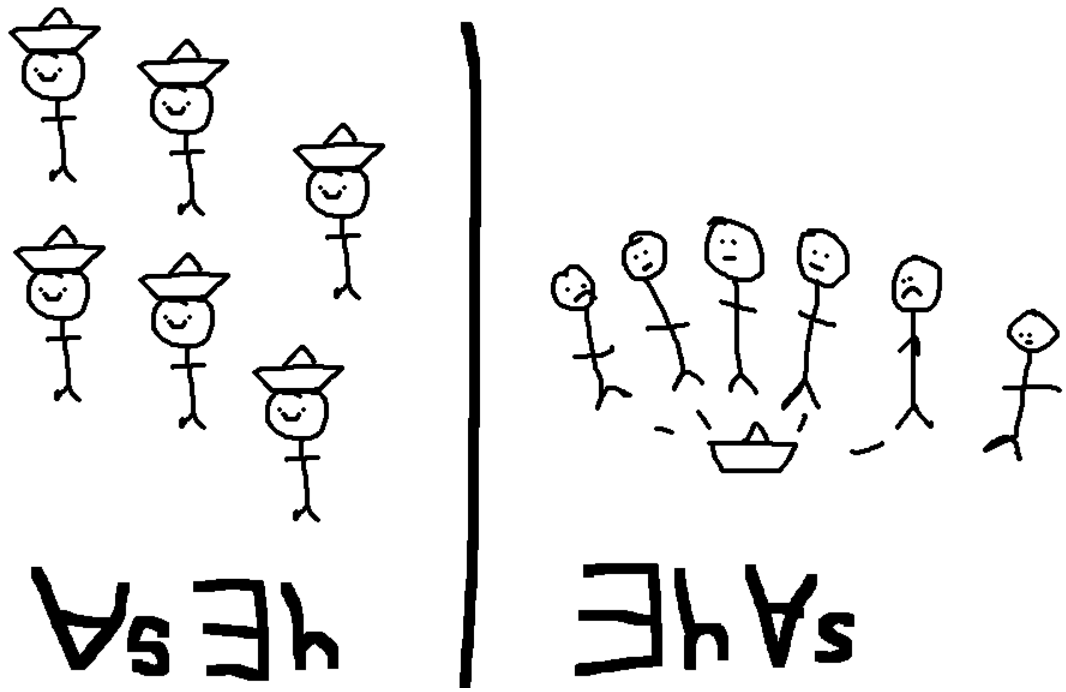
Note that the order of quantifiers does matter. **They do not commute.** Suppose that x, y have universes of booleans. Observe that the following are not equivalent

$$\forall x \exists y[(x \vee y) \wedge (\neg x \vee \neg y)]$$

$$\exists y \forall x[(x \vee y) \wedge (\neg x \vee \neg y)]$$

Consider the semantic difference between "every sailor has a hat" and "some hat has every sailor".

[3]involving both predicates and quantifiers and propositions



7 Negation of Quantifiers

How do you compute the negation of a quantifier? For example let $P(x)$ be the predicate that $P(x)$: "man x is mortal" and the proposition $\forall x P(x)$ to mean that "all men are mortal". The logical opposite of "all men are mortal" may be interpreted as "there is a man who is not mortal". The negation of a universal quantification can be understood as an existential quantification of the opposite!

$$\neg(\forall x P(x)) \equiv \exists x[\neg P(x)]$$

Similarly, the negation of the statement "some man is not mortal" could be understood as "all men are mortal. So

$$\neg(\exists x Q(x)) \equiv \forall x[\neg Q(x)]$$

Consider the propsition $\forall x[x^2 \geq x]$ where the universe of x is integers. Note that this proposition is true. If we compute the negation of it, then

$$\neg(\forall x[x^2 \geq x]) \equiv \exists x \neg[x^2 \geq x] \equiv \exists x[x^2 < x]$$

If your proposition has multiple quantifiers, then we may represent this in a nested manner. For example, $\forall x \exists y P(x, y)$ may really mean $\forall x[\exists y P(x, y)]$, where there is an internal predicate. Negation is handled recursively so that

$$\neg(\forall x \exists y \forall z ... P(x, y, z, ...)) \equiv \exists x \forall y \exists z[\neg P(x, y, z, ...)]$$

Lets compute the negation of the unique existential quantifier. Since

$$\exists ! P(x) \equiv \exists x[P(x) \land \forall y(y \neq x \implies \neg P(y))]$$

Then

$$\neg(\exists ! P(x)) \equiv \forall x[\neg P(x) \lor \exists y(y \neq x \land P(y))]$$

We can read this back in english as either its false for all possible x, or if its true for one x, there exists a distinct y which its also true for. So either it doesn't exist, or if it exists, its not unique.

8 Table

- $\forall x P(x)$ is true when $P(x)$ is true for every x, and is false when there is an x such that $P(x)$ is false.
- $\exists x P(x)$ is true, when there is an x such that $P(x)$ is true, and is false when for every x that $P(x)$ is false.
- $\neg\exists x P(x)$ is understood as "there does not exist an x where $P(x)$ is true", so it is equivalent to "for every x that $P(x)$ is false, or $\forall x \neg P(x)$.
- $\neg\forall x P(x)$ is understood as "It is false that for every x that $P(x)$ is true" which is equivalen to "There is an x such that $P(x)$ is false, or $\exists x \neg P(x)$.