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Lecture 4: Inference

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1 Argument

We have seen that propositional logic is good at removing ambiguity from many parts of natural language, but we have not seen how it can be used to establish truth. The act of deduction is done sequentially, as a sequence of steps. An argument is a sequence of statements which establishes the total, undeniable truth and validity of some statement.

The form of an argument usually begins with a presumed body of knowledge $p_1, p_2, ..., p_k$. Each of these statements consists of the facts, and are presumed true. They are called premises. You wish to *deduce* a *conclusion*, called q. We may represent this as

$$(p_1 \wedge p_2 \wedge ... \wedge p_k) \implies q$$

We conjunct the body of knowledge together because they all must be true. An argument is said to be valid if $(p_1 \wedge ...p_k) \implies q$ is a tautology. We could demonstrate the validity of an argument by writing out a truth table. But note, we actually do not care about situations when any premise $p_1, ..., p_k$ is false. We need to only demonstrate that q follows from when $p_1, ..., p_k$ are all true. Observe that if any premise is false, then the deduction trivially becomes true. Many people[1] do not act illogically, they act logically from wrong premises. The steps of their argument appear correct, but since they assume an invalid premise, then they could "argue" the "truth" of any statement. For an argument to be correct, its premises must also be true.

2 The Rules of Inference

A rule of inference is like a law of thought, in that we may apply it to deduce some statement from a given collection of premises or other deductions. We construct a rule of inference in the following syntax. Let $p_1, ..., p_k$ be the premises, and let q be the conclusion. Then we write

$$\begin{array}{c} p_1 \\ ... \\ p_k \\ \hline \therefore q \end{array}$$

To mean that from premises $p_1, ..., p_k$ we may deduce q. The symbol $\therefore$ means "therefore".

[1] Debate bro's, flat earthers, etc

2.1 Modus Ponens

$$\frac{\begin{array}{c} p \\ p \implies q \end{array}}{\therefore q}$$

If p, and if $p \implies q$, then we may deduce that q is true. It is not too hard to show this is a tautology with a truth table as well.

p	q	$p \implies q$	$(p \implies q) \wedge p$	$((p \implies q) \wedge p) \implies q$	
T	T	T	T	T	
T	F	F	F	T	
F	T	T	F	T	
F	F	T	F	T	

Let p be the proposition corresponding to "Today is Thursday". Let $p \implies q$ correspond to "If today is Thursday, then you will go to class". We may deduce then that q is true, that "you will go to class".

2.2 Modus Tollens

$$\frac{\begin{array}{c} \neg q \\ p \implies q \end{array}}{\therefore \neg p}$$

Loosely, if $p \implies q$, but q never happened, then p didn't happen. If $p \implies q$ corresponds to the proposition "If it is Friday then students wear blue", and $\neg q$ corresponds to "students are not wearing blue", then we may deduce that "It is not Friday" is true.

2.3 Hypothetical Syllogism

$$\frac{\begin{array}{c} p \implies q \\ q \implies r \end{array}}{\therefore p \implies r}$$

If $p \implies q$ corresponds to the proposition "If you make an A on the final, you will pass the class" and $q \implies r$ corresponds to "If you pass the class, then you will graduate on time". We may deduce $p \implies r$, that "If you make an A on the final, you will graduate on time." This also displays that the implication in propositional logic is a transitive relation.

2.4 Disjunctive Syllogism

$$\frac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{\therefore q}$$

If we take $p \vee q$ to mean "Bob brought cake or Alice brought cake" and $\neg p$ to mean "Bob did not bring cake". Then it must be the case that q: "Alice brought cake".

2.5 Addition

$$\frac{p}{\therefore p \vee q}$$

If p : "I like dogs", then $p \vee q$:"I like dogs or cats" is certainly true.

2.6 Simplification

$$\frac{p \wedge q}{\therefore p}$$

If $p \wedge q$ means "I like both dogs and cats" then p : "I like dogs" is true.

2.7 Conjunction

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

If p :"I like dogs" and if q :"I like cats". Then $p \wedge q$:"I like dogs and cats" is true.

2.8 Resolution

$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

If we know it will either rain or snow, and we also know it either won't rain, or hail, then we may deduce that it must either snow or hail. It will rain and hail, or not rain and snow.

3 Examples

Lets give an example of an argument with several steps and several applications of the rules of inference. We demonstrate

$$\begin{array}{c} (\neg p \vee \neg q) \implies (r \wedge s) \\ (r \implies t) \\ \neg t \\ \hline \therefore p \end{array}$$

1. $r \implies t$ (Premise)
2. $\neg t$ (Premise)

3. $\neg r$ (Modus Tollens of 1,2)
4. $\neg r \vee \neg s$ (Addition of 3)
5. $\neg(r \wedge s)$ (DeMorgan's Law)
6. $(\neg p \vee \neg q) \implies (r \wedge s)$ (Premise)
7. $\neg(\neg p \vee \neg q)$ (Modus Tollens of 5,6)
8. $(\neg\neg p \wedge \neg\neg q)$ DeMorgan's law
9. $(p \wedge \neg\neg q)$ Double Negation
10. p Simplification

4 Fallacies

Fallacies can include an incorrect application of correct laws of thought. For example, $((p \implies q) \wedge q) \implies p$ is not a tautology because it may be false when p is false. If $p \implies q$ is to mean "If you get into a car accident, you will die" and q is to mean "You die". You cannot conclude you got into a car accident. You may have died from other methods (perhaps a meteor). This is called the fallacy of affirming the conclusion.

Similarly $((p \implies q) \wedge \neg p) \implies \neg q$ is not a tautology. If you do not get into a car crash, you are not immortal, as you may die of other methods. This is called the fallacy of denying the hypothesis.

5 Quantified Statements

The rules of inference may also apply to those statements which are quantified.

5.1 Universal Instantiation

$$\frac{\forall x P(x)}{\therefore P(c)}$$

If all men are mortal, then Socrates is mortal.

5.2 Universal Generalization

$$\frac{P(c) \text{ for any } c}{\therefore \forall x P(x)}$$

5.3 Existential Instantiation

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some } c}$$

We do not know which c this is true for, only that it is true for some c.

5.4 Existential Generalization

$$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$$

These rules may seem redundant, but they are necessary when you may syntactically need a quantifier or not.