CS 2050 Discrete Mathematics January 18th 2024

Lecture 5: Proof

Lecturer: Abrahim Ladha

1 Why Prove Things?

Our goal with mathematics is to seek truth in all forms. The purpose of proof is to establish the total convincing truth.

It is evident that truth may only be derived and established from other truths. If we wish to demonstrate total certainty of a mathematical statement, then we must make some basic assumptions. These are called *Axioms*

Definition 1.1 (Axiom)**.** An axiom is a statement which may be assumed true without proof.

Different fields of math use different sets of axioms, and the set of axioms you use defines the math you are working in. In real numbers, we have axioms like $ab = ba$, the communitivity of multiplication. Or $a(b + c) = ab + ac$, distributivity. Usually an axiom is so simple, it is impossible to prove it, and there is little debate whether or not an axiom is true. It is so simple that it must be true. In Euclidean geometry, the fourth axiom is "all right angles equal each other".

Definition 1.2 (Theorem)**.** A theorem is a statement which is not an axiom, but has been proven true.

A proof from the axioms involves combining axioms with the laws of thought (themselves axioms) and other proven theorems. A corollary is a theorem which follows some more general theorem. A lemma is a tiny helper theorem used to prove some main theorem. A conjecture is a statement which is unproven. It may be hard to prove, but a mathematician states it hoping someone else may prove it some day in the future.

2 Direct Proof

Definition 2.1 (Even Number)**.** A number is even if it satisfies the predicate

$$Even(n) := \exists k[n = 2k]$$

This is a definition. A number is even if it can be written as two times something. It is even if it can be split in two wholes equally. A number is even if two divides it.

Definition 2.2 (Odd Number)**.** We can equivalently define the predicate

$$Odd(n) := \exists k[n = 2k + 1]$$

Note that an odd number is one which is not even. $Odd(n) \equiv \neg Even(n)$.

Theorem 1. The product of two even numbers is even.

We could write this using the predicate calculus as

$$\forall a \forall b[(Even(a)) \wedge (Even(b)) \implies (Even(ab))]$$

We do not often wish to over detail a theorem in terms of predicates and quantifiers. It can become too cumbersome. Rather, we express them in terms of natural language. This is a relatively simple statement, but already involves two quantifiers, a logical and, and an implication. Statements we wish to prove may be far more complex if written this way. If asked to rewrite a statement into the propositional and predicate calculus, you should be able to. Otherwise, just know it is going on in the background. Now let us prove the theorem.

Proof. Let a be an even number. Then there exists a number k such that $a = 2k$. Let b be an even number. Then there exists a number l such that $a = 2l$. Then $ab = (2k)(2l) = 2(2kl)$. Since we may write ab as two times something, it is even. □

It is polite that the beginning and end of your proof are denoted in some way. In a larger body of text, which may contain more rambling thoughts, you want to make it clear and explicit to the reader where the argument begins and where the argument ends. The reader and writer do a sort of dance or game. Each step is presented to the reader, who digests it, and is convinced of its truth. As the proof concludes, the reader is forced to conclude the what the writer lead them too. Note that this proof actually shows more. It shows that the product of two even numbers is actually divisible by four. Its like, twice as even as normal even number. Doesn't matter. We are tasked with proving that a product of even numbers was even. Were we to conclude that a product of even numbers was divisble by four, it may not be immediate and clear to a reader that is sufficient for it to be even. Lets do some more simple examples.

Theorem 2. The product of an odd number and an even number is even.

Proof. Let a be an even number. Then $a = 2k$ for some number k. Let b be an odd number. Then $b = 2l + 1$ for some number l. Then $ab = (2k)(2l + 1) = 2(k(2l + 1))$. Since we may write ab as two times something, it is even. □

Theorem 3. The product of an odd number and an odd number is odd.

Proof. Let a be an odd number. Then $a = 2k + 1$ for some number k. Let b be an odd number. Then $b = 2l+1$ for some number l. Then $ab = (2k+1)(2l+1) = 4kl+2k+2l+1 = 2(2kl + k + l) + 1$. Since we may write ab as two times something plus one, it is odd. □

Corollary 4. If n is a number and n^2 is odd then n is odd.

Recall a corollary is a tiny theorem following some main one. This actually doesn't directly follow from theorem 3, but from theorem 1 and 3. The product of even numbers is even and the product of odd numbers is odd. So n^2 being odd means that n cannot be even, so it must be odd.

3 Proof by Counter Example

If trying to prove $\forall x P(x)$ false, simply find one example c in which $P(c)$ is false.

Theorem 5. The statement "every positive number is the sum of two numbers which are squared" is false.

We could represent this as $\forall n \exists a \exists b[n = a^2 + b^2]$. To prove that it is false, you simply need to demonstrate an example where it is false.

Proof. Consider $n = 3$. For what values a, b could it be the case that $3 = a^2 + b^2$? Since $2^2 = 4$, we know that each of $a, b < 2$. So a, b can only be $0, 1$. But if we try all combinations of 0^2 and 1^2, we only get the possible values of $0, 1, 2$. So 3 is a counter example to the statement, and it is thus, proven false. □

One of the most famous examples of a counterexample involves the diologue of Diogenes and Plato. Plato, great man and great mind had a school in Athens. He had many students and much recognition. Diogenes was a man who lived in a jar on the outskirts of the city. Guy was committed to the bit, every bit. One day, Plato attempts to establish the definition of a man (as in humanity) Plato asserts that

$$\text{man} \iff \text{featherless biped}$$

All that are humanity are featherless bipeds, and all that are featherless bipeds are man. Plato was interested in a dichotomy and hierarchy of all objects, real or otherwise. To an ancient greek man, the only things he may have seen include some sheep, a mountain, a cloud, etc. Everything is or isn't a biped, and is or isn't featherless. All examples of a biped he may have known had feathers, except man. As the myth goes, Diogenes busts into the amphitheatre, raises a plucked chicken and yells *"Behold! A Man!"*. This is a counter example. Is a plucked chicken a featherless biped? Yes. Is it a man? Certainly not. Then

$$\text{man} \not\Leftrightarrow \text{featherless biped}$$

Diogenes displays this counter example, and proves Plato wrong.

4 Proof by Contraposition

Recall that we proved using a truth table that the contrapositive of an implication was equivalent to it.

$$p \implies q \equiv \neg q \implies \neg p$$

To prove an implication. It may then be easier to prove its contrapositive.

Theorem 6. $3n + 2$ is odd, then n is odd.

Let us try to prove it directly first. We will try to prove that n is odd assuming that $3n+2$ is odd, so $3n+2 = 2k+1$ for some k. Moving terms around, we see that $n = (2k-1)/3$. Its not even clear if that is a number! Lets instead prove the contrapositive.

Figure 1: A Platonic Man

Proof. We prove the equivalent statement that if n is even, then $3n+2$ is even. if n is even, then $n = 2k$ for some number k. If we substitute it into $3n+2$, we get $3n+2 = 3(2k)+2 = 2(3k+1)$ which is even. □

Here, observe that the contrapositive was easier to prove. A direct proof of the theorem may exist, but you want the shortest, clearest proof possible.

Theorem 7. If $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Also difficult to prove directly, but a very useful property of composite numbers. Since n is so general, we don't really have good information about a or b to work with, except that they exist. Lets instead prove the contrapositive.

Proof. Assume that $a > \sqrt{n}$ and $b > \sqrt{n}$. We prove that $ab \neq n$. If $a > \sqrt{n}$ and $b > \sqrt{n}$ then $ab > \sqrt{n}\sqrt{n} = n$. So since $ab > n$, we know $ab \neq n$. □

5 Proof By Contradiction

A proof by contradiction is one of the most versatile techniques, and also may involve some creativity. If you wish to demonstrate some proposition p is true, you can show the negation of the proposition must be absurd. That $\neg p \implies (0 = 1)$. For this reason, it is also called *Reductio Ad Absurdum*

Your proof should always begin soon after stating the theorem. The first sentence of your proof should be an acknowledgement that you are about to perform a proof by contradiction. Traditionally, if you want to prove p, you may begin with "Assume to the contrary $\neg p$". Or sometimes simply "Suppose not". It must be made explicit in some way. You should proceed with deduction applying laws of thought, until you produce *the absurdity.* The absurdity is a statement derived as a consequence of $\neg p$. It aught to be so absurd that the reader will have no choice but to accept that $\neg p$ must be false. The absurdity can take on the form of a negation of a premise or the negation of an axiom. It can take on the form that there is some statement that $p \wedge \neg p$ is true. It must be clear that the absurdity is absurd. If it isn't absurd enough, proceed farther with the proof.

example

Theorem 8. There is no largest number

Proof. Assume to the contrary there was a largest number n. Consider the number $n+1$. We know $n+1$ is a number when n is a number, but $n < n+1$, so n was not the largest number, contradiction. □

The statement of the theorem is obvious, but take note of the setup and syntax.

Theorem 9. If x, y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

Proof. Assume to the contrary that there exists positive real numbers x, y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Then

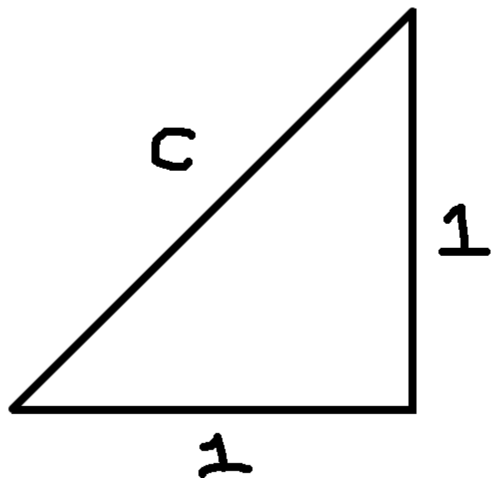
$$\sqrt{x+y} = \sqrt{x} + \sqrt{y} \tag{1}$$
$$x + y = (\sqrt{x} + \sqrt{y})^2 \tag{2}$$
$$x + y = x + 2\sqrt{xy} + y \tag{3}$$
$$0 = 2\sqrt{xy} \tag{4}$$
$$0 = xy \tag{5}$$

By the zero product property, if $xy = 0$, then one of x, y must be zero. This contradicts our assumption that x, y are both positive. □

Note again how we negate the implication here. Recall that $\neg(p \implies q) \equiv p \wedge \neg q$. We phrase this negation as there do exist positive real numbers (p), but $\sqrt{x+y} = \sqrt{x} + \sqrt{y} \quad (\neg q)$.

We finish with one more proof and its legend. Pythagoras is well known for many advancements in mathematics, including the Pythagorean theorem[1] He led a society, a cult maybe, which believed in numerology. They believed that all of nature could be explained by either numbers, or ratio of whole numbers. Today we write $\frac{2}{3}$ and understand it as a "part". They did not. They would have interpretted this as $2:3$, as in two wholes to three wholes, as a ratio. We may eat $\frac{2}{3}$rds of a pie. They would have understood it as two wholes to three wholes. Two pies of three pies. Every number they believed was either whole, or a ratio. The concept of an irrational number was unfathomable to them. Following the Pythagorean theorem grew an essential question. What ratio was the hypotenuse of a right triangle with unit side lengths? How long was the diagonal of a square of side lengths 1?

[1]Even though it had been discovered by others, a few thousand years before him.



Following the Pythagorean theorem, we see that $1^2 + 1^2 = c^2$. For what ratio c could $c^2 = 2$? Today we know that $\sqrt{2}$ can not be rational, it cannot be represented as a ratio of whole numbers. Ancient civilizations thought it might be 577/408 or even 305470/216000, but these are simply approximations. Pythagoras could not comprehend that an irrational number could exist, since it contradicted his view of nature. A student of his, was able to demonstrate that not only do irrational quantities exist, but $c = \sqrt{2}$ must be irrational.

Theorem 10. The number $\sqrt{2}$ is irrational.

Proof. Assume to the contrary that $\sqrt{2} = m/n$ for m, n numbers in reduced form. The numbers m, n do not share any factors, the ratio has been simplified. Certainly every rational number can be written in such a reduced form. Since it is reduced, we know both m, n cannot be even, so one must be odd. We may write

$$\sqrt{2} = \frac{m}{n} \tag{6}$$
$$\sqrt{2}n = m \tag{7}$$
$$(\sqrt{2}n)^2 = m^2 \tag{8}$$
$$2n^2 = m^2 \tag{9}$$

Since we may write m^2 as two times something, it must be m^2 is even. Since the square of an odd number is always odd, then m must also be even. So $m = 2k$ for some k. Then

$$2n^2 = m^2 \tag{10}$$
$$2n^2 = (2k)^2 \tag{11}$$
$$2n^2 = 4k^2 \tag{12}$$
$$n^2 = 2k^2 \tag{13}$$

Since we can write n^2 as two times something, n^2 is also even, so we know that n must also be even. But how can both m, n be even? We assumed they were both reduced! If they are both even, they are not reduced, as they share the common factor of two. A contradiction. □

I will quote a popular book[2] to finish the myth: "For Pythagoras, the beauty of mathematics was the idea that rational numbers (whole numbers and fractions) could explain all natural phenomena. This guiding philosophy blinded Pythagoras to the existence of irrational numbers and may even have led to the execution of one of his pupils. One story claims that a young student by the name of Hippasus was idly toying with the number $\sqrt{2}$, attempting to find the equivalent fraction. Eventually he came to realise that no such fraction existed, i.e. that $\sqrt{2}$ is an irrational number. Hippasus must have been overjoyed by his discovery, but his master was not. Pythagoras had defined the universe in terms of rational numbers, and the existence of irrational numbers brought his ideal into question. The consequence of Hippasus' insight should have been a period of discussion and contemplation during which Pythagoras ought to have come to terms with this new source of numbers. However, Pythagoras was unwilling to accept that he was wrong, but at the same time he was unable to destroy Hippasus' argument by the power of logic. To his eternal shame he sentenced Hippasus to death by drowning. The father of logic and the mathematical method had resorted to force rather than admit he was wrong. Pythagoras' denial of irrational numbers is his most disgraceful act and perhaps the greatest tragedy of Greek mathematics. It was only after his death that irrationals could be safely resurrected."

[2] Fermat's Enigma by Simon Singh