CS 2050 Discrete Mathematics	February 1st 2024

Lecture 8: Elementary Set Operations

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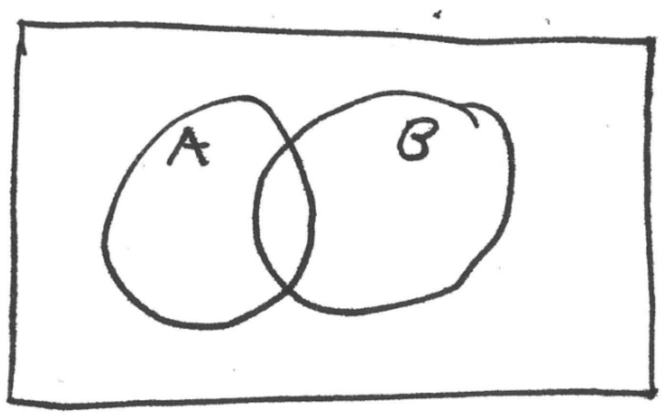
Today we discuss common set operations.

1 Venn Diagrams

A Venn diagram is a picture which looks like this



We use it to show the relationship between two sets A, B. What is in one, not the other, in both, or in niether.

2 Union

Definition 2.1. Let A, B be any two sets from the same universe of discourse Ω. We define $A \cup B$ to be

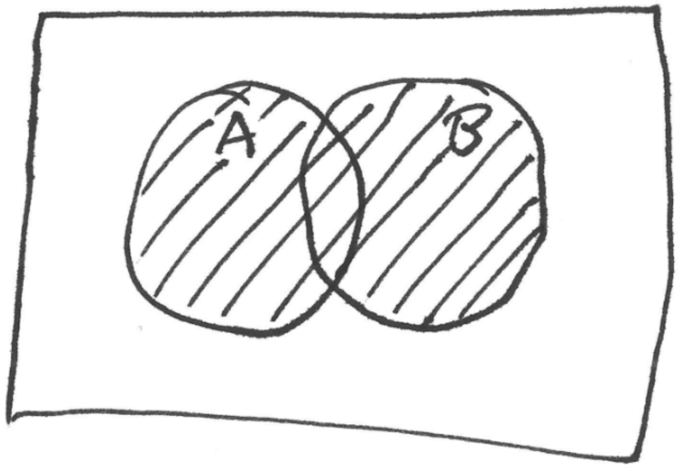
$$A \cup B = \{x \in \Omega \mid x \in A \text{ or } x \in B\}$$

.

The union of two sets is a new set containing all the elements of both parts.

- $\{1, 2, 3, 4\} \cup \{1, 5, 7, 8\} = \{1, 2, 3, 4, 5, 7, 8\}$
- $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$
- $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$

It doesn't matter that A, B may contain some of the same elements, one will be in $A \cup B$. Also, observe that if $A = B$ then $A \cup B = A = B$. And if $A \subsetneq B$ then $A \cup B = B$.



3 Intersection

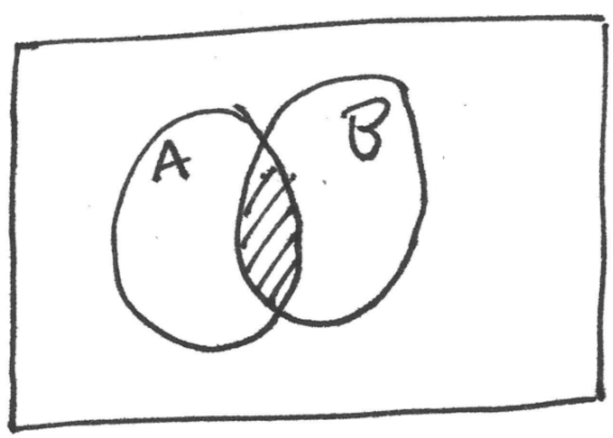
Definition 3.1. Let A, B be any two sets from the same universe Ω. We define $A \cap B$ to be

$$A \cap B = \{x \in \Omega \mid x \in A \text{ and } x \in B\}$$

The intersection of two sets contains exactly and only the elements that are in both.

- $\{1, 2, 3, 4\} \cup \{1, 5, 7, 8\} = \{1\}$
- $\mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$
- $\mathbb{Q} \cap \mathbb{I} = \emptyset$

Note that if $A = B$ then $A \cap B = A = B$ and if $A \subseteq B$ then $A \cap B = A$. Also note that $|A \cap B| \leq \min(|A|, |B|)$.



4 Complement

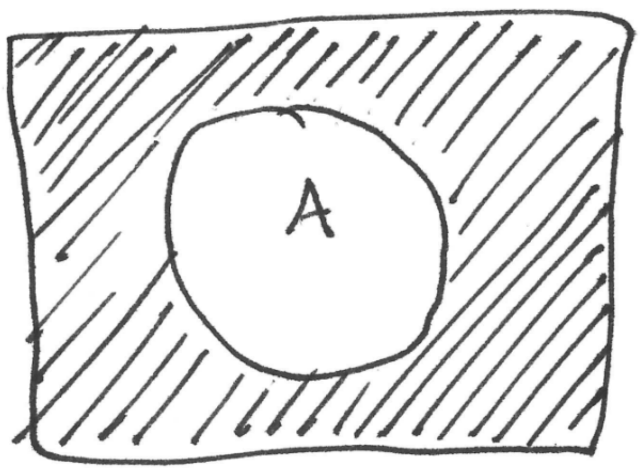
Definition 4.1. If $A \subseteq \Omega$ some set, then we define the complement of A written as $\overline{A}$ as

$$\overline{A} = \{x \in \Omega \mid x \notin A\}$$

You may also see this written as $A^{\complement}$.

- $\overline{Q} = \mathbb{I}$
- If $\Omega = \mathbb{N}$ and $A = \{2, 3, 4\}$ then $\overline{A} = \{0, 1\} \cup \{5, 6, ...\}$

observe that $\overline{\overline{A}} = A$. Also $x \in A \iff x \notin \overline{A}$



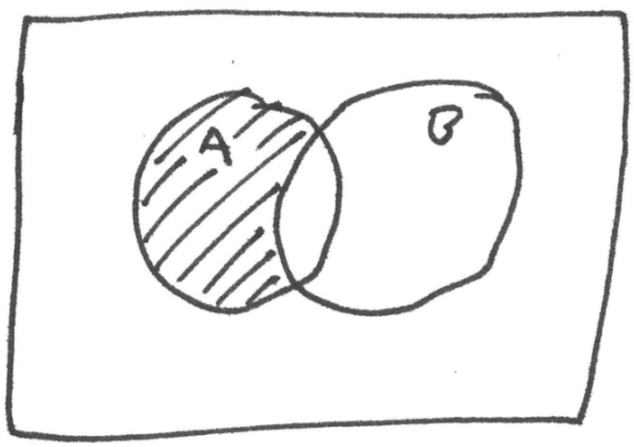
5 Difference

Definition 5.1. The difference of two sets A, B, written $A \setminus B$ is defined as $A \setminus B = \{x \in \Omega \mid x \in A \text{ and } x \notin B\}$.

For $A \setminus B$, you keep everything in A but take out everything B, if there is anything to take out. Think of it like A except B. ALl reptiles except lizards.

- $\mathbb{Z} \setminus \mathbb{N} = \{..., -2, -1, 0\}$
- $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$

Also note that $A \setminus B = A \cap \overline{B}$.



6 DeMorgan's Law

Theorem 1.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Although you can prove it with identities, we prove it using a double set containment.

Proof. We proceed by a double set containment.

- ($\subseteq$)

$$\begin{aligned} x &\in \overline{A \cup B} \\ x &\notin A \cup B \\ x \notin A \text{ and } x &\notin B \\ x \in \overline{A} \text{ and } x &\in \overline{B} \\ x &\in \overline{A} \cap \overline{B} \end{aligned}$$

- ($\supseteq$)

$$\begin{aligned} x &\in \overline{A} \cap \overline{B} \\ x \in \overline{A} \text{ and } x &\in \overline{B} \\ x \notin A \text{ and } x &\notin B \\ x &\notin A \cup B \\ x &\in \overline{A \cup B} \end{aligned}$$

□

7 Identities

- $A \cap \Omega = A$ **Identity**
- $A \cup \emptyset = A$

- $A \cup \Omega = \Omega$ **Domination**
- $A \cap \emptyset = \emptyset$

- $A \cup A = A$ **Idempotent**
- $A \cap A = A$

- $\overline{\overline{A}}$ **Complementation**

- $A \cup B = B \cup A$ **Communitivity**
- $A \cap B = B \cap A$

- $A \cup (B \cup C) = (A \cup B) \cup C$ **Associativity**
- $A \cap (B \cap C) = (A \cap B) \cap C$

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **Distributive**
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ **DeMorgan's**
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- $A \cup (A \cap B) = A$ **Absorption**
- $A \cap (A \cup B) = A$

- $A \cup \overline{A} = \Omega$ **Complement**
- $A \cap \overline{A} = \emptyset$

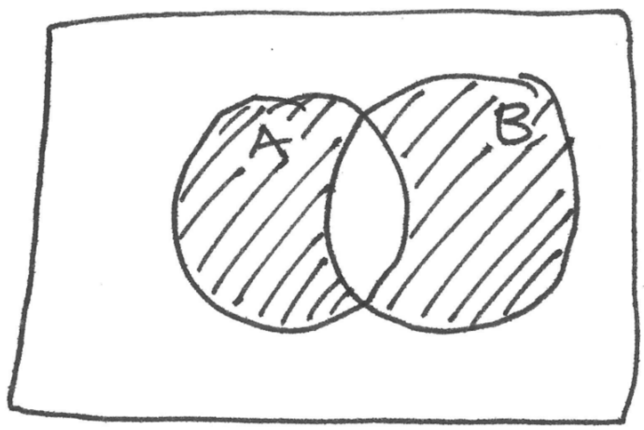
8 Symmetric Difference

Definition 8.1. The symmetric difference of two sets A, B, usually written $A \oplus B$ or $A \triangle B$ is defined as

$$A \triangle B = \{x \in \Omega \mid x \in A \text{ or} x \in B \text{ but not both}\}$$

It can be written in many equivalent ways.

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cap \overline{B}) \cup (\overline{A} \cap B)$$



9 Indexed Collections

An indexed collection of sets is simply a way to notate many many sets when venn diagrams get too big. We have described operations on two sets A, B. What if you had twenty or a hundred sets? What if you had infinitely many sets? Let us do a few examples

Let $A_i = \{i, i+1\}$. Then

$$\bigcup_{i=1}^{5} A_i = \{1,2\} \cup \{2,3\} \cup \{3,4\} \cup \{4,5\} \cup \{5,6\} = \{1,2,3,4,5,6\}$$

Lets do an example with intervals

$$\bigcap_{i=n}^{\infty} \left[0, \frac{1}{n}\right) = \{0\}$$

This is an infinite intersection of intervals, each containing infinitely many elements. The intersection though only contains one element. Let us prove it.

Proof. Assume to the contrary that the intersection is not simply the set containing zero. Since zero is in the intersection, and any element of the intersection must be positive, suppose that some element $x > 0$ is in the intersection. Then we know that there is a k such that $\frac{1}{k+1} < x < \frac{1}{k}$. So $x \notin [0, \frac{1}{k+1})$. But then x is not in the intersection, contradiction. □

10 Partition

Definition 10.1. A partition of some set S is a collection of sets A, B such that $A \cup B = S$ and $A \cap B = \emptyset$

A set can be partitioned into multiple sets. A parition of some set S is a a collection of sets $A_1, A_2, ...$ such that

$$S = \bigcup_{i=1}^{\infty} A_i$$

and if $i \neq j$ then $A_i \cap A_j = \emptyset$

For example, we may partition $\mathbb{N}$ into two sets, one containing evens, and one containing odds. $\mathbb{N} = E \cup O$. We may also partition it into infinitely many subsets each containing one element.

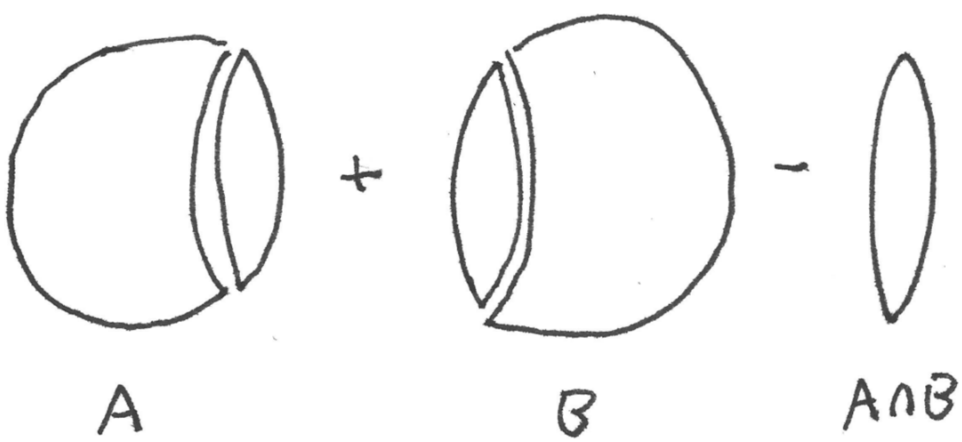
$$\mathbb{N} = \bigcup_{i=0}^{\infty} \{i\}$$

11 Inclusion Exclusion

Theorem 2. If A, B are any two sets, then

$$|A \cup B| + |A| + |B| - |A \cap B|$$

We don't yet have the tools to prove this in the general case, but we can allude to it pictorally. You may want to think that $|A \cup B| = |A| + |B|$ but actually this double counts elements that in both A, B. So you must subtract off the intersection. Visually:



Although obvious visually, we will demonstrate using the laws of thought that

$$A = (A \setminus B) \cup (A \cap B)$$

.

Proof.

- $A \equiv$
- $A \cap \Omega \equiv$ Identity
- $A \cap (B \cup \overline{B}) \equiv$ Complement
- $(A \cap B) \cup (A \cap \overline{B}) \equiv$ Distributivity
- $(A \cap B) \cup (A \setminus B)$ not a law just definition

□