CS 2050 Discrete Mathematics February 1st 2024

Lecture 9: Functions

Lecturer: Abrahim Ladha

1 Relations

Given two elements, maybe of the same set maybe of different sets, there are many ways we can relate them.

Definition 1.1. For any sets A, B we define a relation R to be a subset of $R \subseteq A \times B$

We write aRb for $a \in A$ and $b \in B$ to mean a relates to b. The symbol R here is generic, and is usually meant as an operator. Inequality of real numbers is a relation:

$$\text{“} \leq \text{”} = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \leq b\}$$

There are many more interesting properties of relations we may go into at a different time.

2 Functions

A function is a kind of relation.

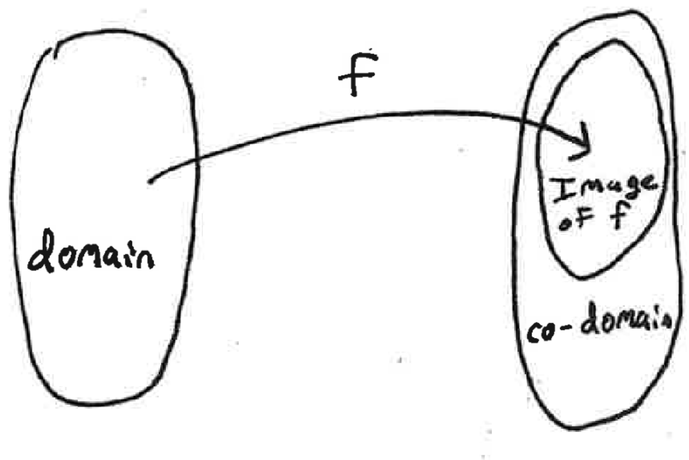
Definition 2.1. We say a function from two sets A, B written $f : A \to B$ is a relation $f \subseteq A \times B$ with the property that every element a of A corresponds to at most one pair (a, b) of f. If $f(a) = b$ and $f(a) = c$, then $b = c$. If $(a, b), (a, c)$ are two ordered pairs, then if f is a function then $b = c$.

You should have an excellent understanding of some common and popular functions. We can write them as set builder notation

- $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = e^x\}$ is usually written as $f(x) = e^x$
- $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$ is usually written as $f(x) = x^2$

These you have associated as an input and output, but you can formally describe a function as a subset of a cartesian product. Using sets, we can define functions as a kind of set.

- The domain of a function $f : A \to B$ is A, the set where its input is defined.
- The co-domain of a function $f : A \to B$ is B, the set where its output is defined.
- The image of a function is defined as a subset of the co-domain which has values mapped to.
- For example, if $f(x) = x^2$, then the domain of f is $\mathbb{R}$, the codomain is $\mathbb{R}$, but the image is $\{x \in \mathbb{R} \mid x \geq 0\}$. Not every element of the codomain gets mapped to.



2.1 Equality of two functions

Definition 2.2. Two functions $f : A \to B$ and $g : A \to B$ are said to be equal if

$$\forall a \in A[f(a) = g(a)]$$

They map the same elements to the same elements. Two functions are not equal if they only have the same domain and co-domain. They must map the same elements to the same elements.

Definition 2.3. A function is said to be total if it maps every element in its domain. A function is partial if there is some element in its domain which it does not map.

This is less important. If you have a partial function, you may simply consider its domain to be the subset of elements which it maps. In some contexts, there is an assumption that every function is total. We shall make this assumption in class.

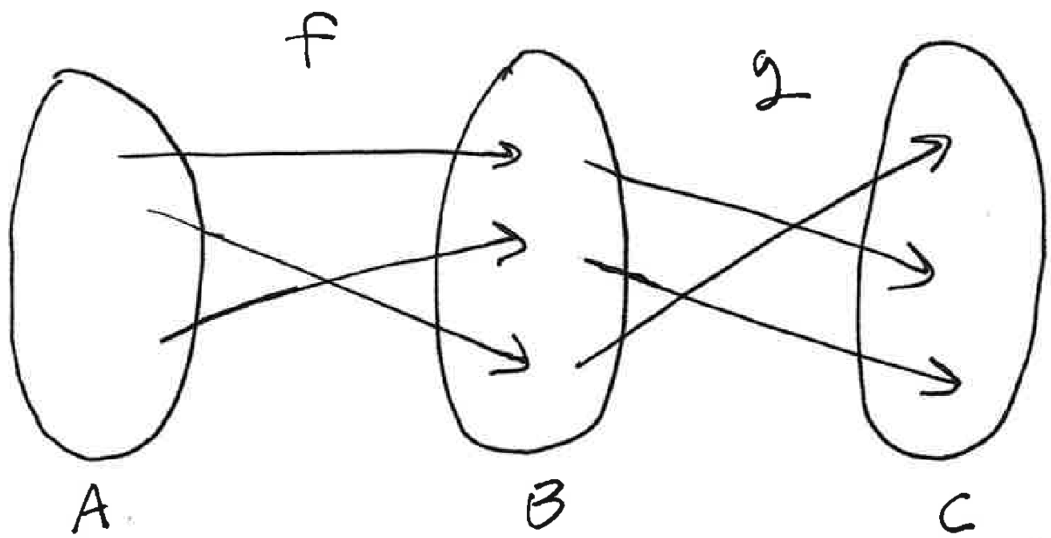
This formulation of what is or isn't a function is relatively modern. Functions used to be too restrictive in their definition, but the generality of this definition of function allows for some interesting examples. For example

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Does this function have a derivative? Is it even continuous? Does it even have an area under its curve? Or an average of its values within some interval? Functions can be not nice, and have some very interesting properties. The Descartian idea of a function was something which looked nice and you could plot in the cartesian plane. I claim that you could not even attempt to plot this function. Even though this is not a nice function, $|2f(x) - 1| = 1$, a constant function. Extremely nice.

3 Combining functions

Definition 3.1. Let f, g be two functions $f : A \to B$, $g : B \to C$. We define the composition of f, g as $= g(f(x))$.



f returns an element of B, which is part of the domain of g, so g then maps this element to some element of C. The domain of $g \circ f$ is A but the codomain of $g \circ f$ is C.

The set definition of a function allows you to define multiple arguments. For example, if you want to define a function that takes two numbers and outputs a third, you may write $f(a, b) = c$. Rather than defining two domains of this function somehow, it only has one domain, $\mathbb{N} \times \mathbb{N}$. The input is not two distinct numbers, rather an element of the cartesian product $(a, b) \in \mathbb{N} \times \mathbb{N}$.

4 Bijectivity

Definition 4.1. We say a function f is injective, or one-to-one if $a \neq b \implies f(a) \neq f(b)$. Element distinctness in the domain implies distinctness after mapping into the co-domain.

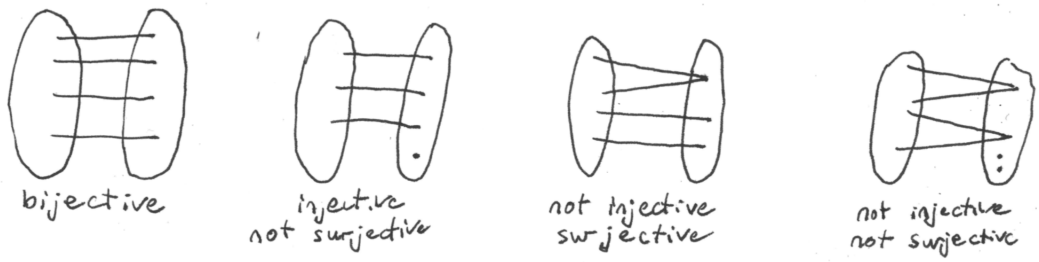
To prove a function to be injective, you usually use the contrapositive. $f(a) = f(b) \implies a = b$.

Definition 4.2. We say a function f is surjective, or onto if $\forall b \in B, \exists a$ such that $f(b) = a$.

A function is surjective if there is no element in the co-domain which remains unmapped. Nothing is left behind. A function is surjective if its image is equal to its co-domain.

Definition 4.3. A function is bijective if and only if it is injective and surjective.

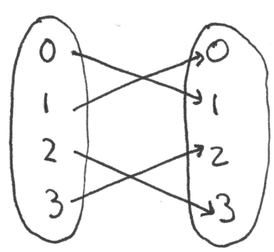
When you think of a bijection, you should think of the following picture. A bijection is a perfect correspondence between the domain and co-domain.



Theorem 1. The function $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ with $f(x) = 3x + 1 \pmod 4$ is a bijection

Proof. We can prove a function to be bijective by proving it is injective and surjective. We first prove it is injective. Let $f(a) = f(b)$. We prove $a = b$. By the definition of f, we know $f(a) = f(b)$ means $3a + 1 \equiv 3b + 1 \pmod 4$. By laws of modular arithmetic, we know that $3a \equiv 3b \pmod 4 \implies a = b$.

Next we show it is surjective. For such a simple function, we may write out each value, and note all four elements of the co-domain have some elment of the domain mapping to them.



Since we have shown that f is injective and surjective. It is bijective. $\square$

Lets do a harder proof of bijectivity.

Theorem 2. The function $\mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{3\} = 3x/(x-2)$ is a bijection.

Proof. We prove f is a bijection by proving it is injective and surjective. Suppose that $f(a) = f(b)$. We will prove $a = b$. If $f(a) = f(b)$ then

$$\frac{3a}{a-2} = \frac{3b}{b-2} \implies 3a(b-2) = 3b(a-2) \implies 3ab - 3a = 3ab - 3b \implies 3a = 3b \implies a = b$$

Next we show it is a surjection. Let r be any element of the co-domain. Consider the element of the domain $x = \frac{2r}{r-3}$. We show that $f(x) = r$.

$$f(x) = 3\frac{2r}{r-3}\frac{1}{2r/(r-3)-2} = \tag{1}$$

$$3\frac{2r}{r-3}\frac{}{(2r-2r+6)/(r-3)} = \tag{2}$$

$$3\frac{2r}{r-3}\frac{r-3}{(2r-2r+6)} = \tag{3}$$

$$\frac{6r}{6} = r \tag{4}$$

□

You usually have to do a proof three times. First to figure out if its true, second to work out some of the details and structure of the proof, and a third time formally. This proof is the third, final version, and it does not show how we were able to get the proof itself. To show surjectivity, behind the scenes, we computed the inverse of the function to get $x = 2r/(r-3)$. Once we had this, we could procede with the proof of surjectivity normally.

5 Monotonic Functions

Definition 5.1. A function $f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing if

$$x \leq y \implies f(x) \leq f(y)$$

A function is said to be strictly monotonically increasing if

$$x < y \implies f(x) < f(y)$$

Think of a monotonically increasing function as one whos plot does not go back down, it is always going up, or staying flat.

Theorem 3. $f(x) = x^2$ is strictly monotonically increasing.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then $x^2 < xy$ and $xy < y^2$. So $x^2 < y^2$ as desired. □