Lecture 13: Foundation of Mathematics

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1 A Motivation from Geometry

Recall last time we had a lecture on pure mathematics, countability, and set theory. The lecture before that was kind of pseudo-philosophical on the Church-Turing thesis. The lecture before that one was on engineering, programming and understanding the Turing machine. To keep the trend of not having one, today's lecture will be on history.

We will go from 300 BC to 1936. We begin of course, with the Greeks. Around 300 BC, Euclid wrote "The Elements", several treatises in geometry. It is one of the most influential texts of all time. It established mathematics as a deductive rather than empirical science. It has been in print for millennia and comes second only to the bible. Just because calculus wasn't invented yet didn't mean you didn't have math class. You used to have to take a three course series on classical geometry.

Recall that a theorem is some statement proved. From what? Other theorems? Not quite. There is some flow of implications in this giant tree of knowledge. Follow back to eventually reach some root: The axioms.

Definition 1.1 (Axiom)**.** An axiom is a statement that needs no proof. It may be assumed to be true.

It is usually so trivial to be anything but true. For example, consider the associativity of addition. This is can be expressed as

$$\forall a, b, c[(a + b) + c = a + (b + c)]$$

If asked to prove this, you could simply cite the axiom and perform no deduction.

Definition 1.2 (Theorem)**.** A theorem is some statement which is deduced from the axioms.

A lemma is a theorem which is prior to a main theorem. A corollary is a theorem which comes after some main theorem, like a little dessert.

Definition 1.3 (Proof)**.** A proof is an application of axioms with the "rules of deduction" which are themselves axioms. Given axiomatic system AS, we may write that T is a theorem of AS as $AS \vdash T$. There exists a proof of the statement T from the axioms of AS.

Definition 1.4 (Model)**.** A model is a structure to satisfy some set of axioms. Under given relational symbols, you can think of a model as its true statements, its theorems, relative to that set of axioms.

Euclid defined[1] his first five axioms, or postulates as follows:

[1]Of course, he did it in ancient Greek. Here they have been modernly rephrased. Look up Playfair's axiom if you are interested in this rephrasing.

1. any two points may be connected by a line segment.
2. any line segment may be extended infinitely in both directions.
3. for any point and radius, there exists a circle.
4. all right angles equal each other
5. given a line l and a point p, not on that line, there exists exactly one line through p parallel to l.

All of the axioms for Euclid's elements are a model for what we now call Euclidean geometry. From the axioms, you can prove things like: every square has four equal right angles, the sum of the interior angles of a triangle is $180°$, if a triangle has two equal angles it has two equal sides, and so on.

Euclid's elements have nothing to do with geometry. It is about rigorous and systematic thinking. It is nothing more than a by-product of the school of thought that Euclid and other Greeks came from. Plato's Theory of Forms asserts that ideas are the supreme achievement of human beings. They are a refined, pure reflection of the capability of the human mind. There exists the Real: the material, empirical, measurable, and approximate world. There also exists the Ideal: one of concepts and thought. The Real and Ideal certainly have a duality[2]. This school of thought asserts that the world, the materiality, is shaped by some things prior to it, the immaterial. When I draw a triangle on the board, understand this exists no where except in your mind. No Real triangle you can form from sticks, or by drawing in the sand can ever reach the precision of the Ideal triangle. However, by studying the Ideal, it may reveal to you something about the Real. To understand the material, you only need to understand the non-material. Abraham Lincoln famously used the Elements to train as a lawyer.

> At last I said, 'Lincoln, you never can make a lawyer if you do not understand what demonstrate means'; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.

In 1854 in an unpublished note, he used this rigorous thinking to assert abolition.

> If A. can prove, however conclusively, that he may, of right, enslave B. — why may not B. snatch the same argument, and prove equally, that he may enslave A? – You say A. is white, and B. is black. It is color, then; the lighter, having the right to enslave the darker? Take care. By this rule, you are to be slave to the first man you meet, with a fairer skin than your own. You do not mean color exactly? – You mean the whites are intellectually the superiors of the blacks, and, therefore have the right to enslave them? Take care again. By this rule, you are to be slave to the first man you meet, with an intellect superior to your own. But, say you, it is a question of interest; and, if you can make it your

[2]Maybe are better known by other names to you, theory and practice?

interest, you have the right to enslave another. Very well. And if he can make it his interest, he has the right to enslave you.

Definition 1.5 (Axiomitization)**.** Axiomitization is a process in which one creates formal rules to capture basic intuitive premises.

We do not exist in this world as actors carrying out some known rules, even though the natural world is governed by rules. Part of the job of the scientist is to discover what some of the rules are. A physicist observes a ball bouncing, and produces an equation to explain a phenomena. The ball doesn't know this equation exists, and doesn't bounce because the equation says it does. Axiomitization is a process in which you take samples of the human experience, and attempt to construct a set of axioms such that the model which satisfies the set of constructed axioms could only be the model of our own lived experiences. Truths within the model should mirror reality. We treat associativity of addition as an axiom because our use of the combinations of quantity of items appears independent of the order we perform the combination in. Millennia was spent trying to refine Euclid's Elements, to show they were only as good and simple as necessary.

Definition 1.6 (Independence)**.** A axiom A is independent relative to a set of axioms AS, if there is no proof of A from AS. That is, $AS \nvdash A$. A set of axioms is said to be independent if no axiom can be proven from the others.

You may relate the independence of a set of axioms with the independence with respect to a basis of a vector space. If an axiom could be proved from the others, then it need not be an axiom. Remove it, and simply take it as a theorem. The fifth axiom took a lot attention as if it was the first unobvious one. The first four are just definitions. Let PP be the parallel postulate and EE the axioms of Euclid's Elements.

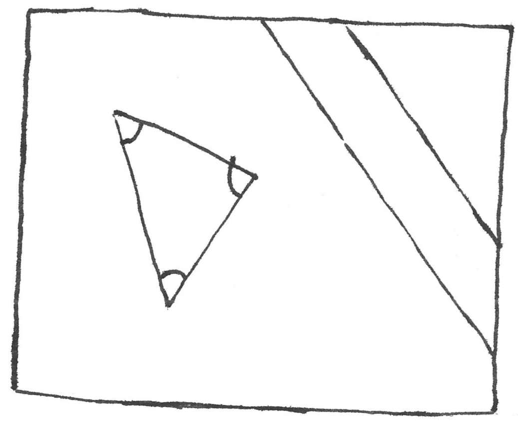
First, is the fifth postulate even necessary as an axiom? Need it be an axiom? Or can it be taken as a theorem. Notationally, we would present this as $(EE - PP) \vdash PP$? We now know the fifth postulate is independent, so this is impossible.

So you cannot take the fifth postulate as a theorem, but do you still need to take it as an axiom? How much of the truth of the model is dependent upon it? There were attempts to prove that $(EE - PP + \neg PP) \vdash 0 = 1$. That is, if you removed the axiom and assumed its negation, you would produce a contradiction.

Definition 1.7 (Consistency)**.** A set of axioms AS is consistent if $AS \nvdash (0 = 1)$.

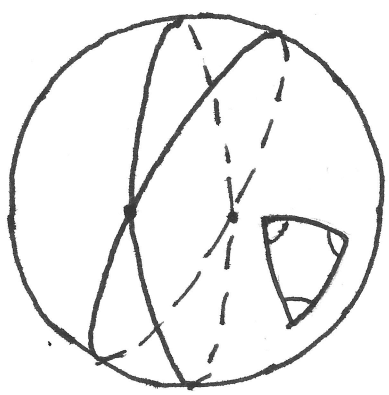
If you can prove a falsehood, then all statements are true and false simultaneously, and the model is uninteresting. If assuming a negation of the parallel postulate led to an inconsistency, then certainly it is strong evidence in favor of taking it as an axiom (unnegated). This should be true for any usable and necessary axiom, but they discovered something insane. Taking the negation doesn't produce an inconsistency, instead it produces two different consistent models!

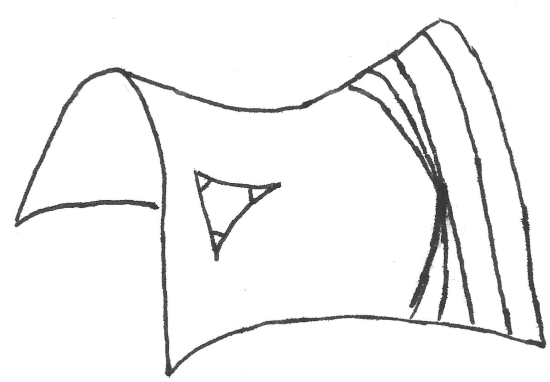
Recall the parallel postulate says "Given a line and a point not on that line, there exists **exactly one** line through the point parallel to the line."



The model of Euclidean geometry appears as such. The interior angle sum of triangles is exactly 180 degrees, and other classic theorems of Euclidean geometry you are familiar with. If the number of parallel lines through the point equals one, we live in a flat world, the Euclidean plane. Any other parallel lines would be equivalent since they intersect at all points. In fact, the parallel postulate is exactly equivalent to the existence of any triangle with interior angle sum of 180 degrees.

We may take the negation of the parallel postulate into two cases. We change "exactly one" to "greater than one" or "less than one". We are changing the number of possible parallel lines from $= 1$ to > 1 or < 1.





If the number of parallel lines is less than one, as in there do not exist any parallel lines anywhere ever, we are in the model of spherical or ellipsoid geometry. We are embedded onto an egg, or globe. The lines on a sphere are only the great circles, using Euclid's definition, the smaller circles, laterals other than equators, are curves. The interior angle sum of a triangle in this model of geometry is $\gneqq 180$, and approaches 180 as you take the limit of the area of the triangle to zero. Here is a triangle with interior angle sum of 270. Start at the north pole, and go straight down to the equator. Then walk along a quarter of the equator. Then go straight back up to the north pole. You will have outlined an eighth of the surface area of the sphere and have a triangle with three right angles! In this model, all the classic theorems of geometry are slightly perturbed, but the model is not inconsistent.

If the number of parallel lines were greater than one, then we exist not on the plane, but on a saddle, or a pringle. In this model, a line is a hyperbola. There exists many "parallel" lines through the point which do not intersect our line. The interior angle sum of a triangle is always $\lneq 180$. This is also called hyperbolic, or Lobachevskian[3] geometry. All the classic theorems of geometry, are again, slightly perturbed, but still not inconsistent.

Note that these are consistent models. They cannot prove $0 = 1$ but all the theorems which these models derive are slightly different. The fact that there exists more than one consistent model of geometry is foundationally shaking. What could this mean for the nature of truth itself? We formulated Euclidean geometry following our empirical experiences in the Real, measuring angles and generalizing our observations. Maybe the Real could follow these models instead? Who can say? How can we know that every time we have measured the angles of a triangle, it hasn't technically been $180 + \varepsilon$ this whole time? We do live on a sphere after all.

It was Kant, Spinoza, and many others who had deep regard for geometry as an example asserting their philosophies. Geometry, as we experience it, is Kant's example of knowledge which is not learned, but inherent to us. They implicitly assumed that all of geometry was only Euclidean geometry. The existence of unequivalent, yet consistent models of geometry was shattering. It motivated the next several generations of work into what constitutes truth and knowledge. More importantly, this sparked more serious concerns about the foundation of all of mathematics, not just geometry. Who is to say alternative, consistent models of arithmetic and functions don't also exist?

2 Analytic Philosophy

Aristotle was perhaps the first to remark on the axiomatic method. First you begin with some axioms, as without assumptions, nothing can be derived. New knowledge can only be derived from old knowledge by the application of rational deduction. Centuries later, Leibniz made an important remark on this formalization of logic. He noted that ideas were compounded from some "alphabet of human thought". He also remarked that complex ideas proceed from these by a process analogous to arithmetical multiplication.

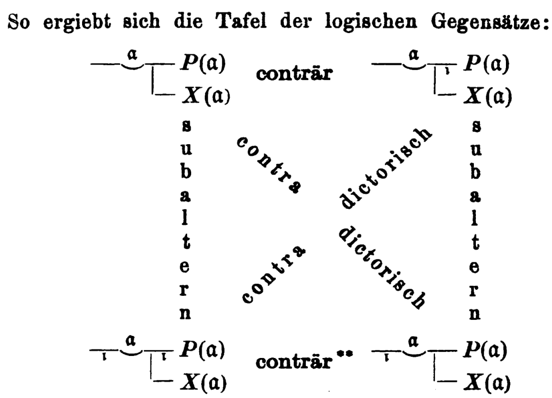
> It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines, we could do in all matters insofar as they are subject to reasoning all that we can do in arithmetic and geometry. For all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus.

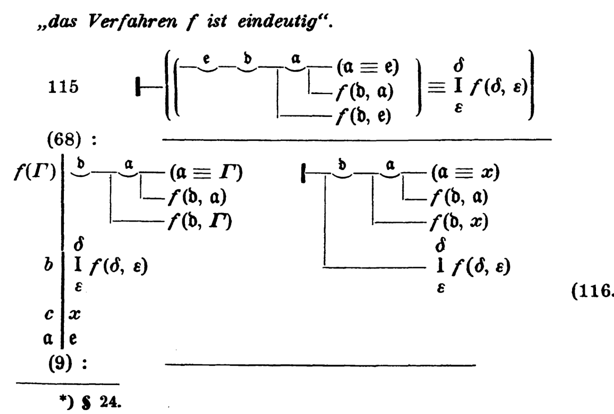
It was Gottlob Frege who tasked himself with carrying out Leibniz's vision. Begriffsschrift is his seminal work. It identifies as "a formal language modeled on that of arithmetic, for pure thought". He gave the most modern definition of logic, formalizing many parts into what we use today. These include variables, quantifiers, implications, and so on. Each of these are formalizations of the intuitive concepts of our own rules of deduction. Although

[3] Lobachevsky is equally remembered for his model of geometry, as he was for accusations and rumors of plagarism. `https://www.youtube.com/watch?v=gXlfXirQF3A`

he used quite a different notation than we use today, we still use many of the axioms he first described.

- $A \implies \neg\neg A$
- $C = D \implies f(C) = f(D)$
- $(A \implies B) \wedge A \implies B$[4]





Also importantly, he defined the following two axioms.

Definition 2.1 (Axiom of Existentionality).

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$$

It simply defines the equality relation among sets. Two sets are equal if they contain the same elements.

Definition 2.2 (Axiom of Unrestricted Comprehension).

$$\exists y \forall x [x \in y \Leftrightarrow \Phi(x)]$$

Here, Φ is any statement, a predicate written over the formal symbols. The axiom of unrestricted comprehension asserts that if there is a description of a kind of object, then there exists a set of those objects. For example we may apply the axiom of unrestricted comprehension to deduce that a set of all primes exists.

$$Prime(x) = \neg\exists z[(z \leq x) \wedge \neg(z = 1) \wedge \neg(z = x) \wedge (z|x)] \wedge (x > 1)$$

3 A set of Axioms for the Natural Numbers

In 1889, Giuseppe Peano gave us a set of axioms to describe the structure of the natural numbers. Frege gave an axiomitization of a theory of sets, while Peano gave an axiomitization of a theory of numbers, and they worked independently. The following are the Peano axioms.

[4] This is called modus ponens

- $\exists 0$ Zero exists and is a number
- $\forall x[x = x]$ Equality is reflexive
- $\forall x \forall y[x = y \iff y = x]$ Equality is symmetric
- $\forall x \forall y \forall z[(x = y) \land (y = z) \implies (x = z)]$ Equality is transitive
- $\forall x \forall y[(x \in \mathbb{N}) \land (x = y) \implies y \in \mathbb{N}]$ The naturals are closed under equality
- $\forall n[n \in \mathbb{N} \implies S(n) \in \mathbb{N}]$ The naturals are closed under sucession with $S(n) = n + 1$
- $\forall m \forall n[S(n) = S(m) \implies n = m]$ S is injective
- $\neg \exists n[S(n) = 0]$ No number precedes zero.

From just these axioms, you may build arithmetic and number theory. The operations of addition and multiplication are inductively defined relations following from just these axioms.

4 A Foundation from a Theory of Sets

Numbers, although incredibly powerful, are not powerful enough to model other important parts of mathematics, other than themselves. It was quickly discovered that sets are extremely powerful in modeling many other parts of mathematics, functions, geometry, and so on. Axiomatically given sets, you may construct the numbers, but axiomatically given numbers, you may not construct sets. To prove that you can construct the numbers given sets, you need to prove Peano's axioms as theorems from your axioms. Lets prove $1 = 1$ using Set theory. Assume we have Frege's axioms of sets, and not Peano's axioms of numbers. We first need to construct the numbers. An ordinal is a set which we pretend is a number. First, we define zero. We apply the axiom of unrestricted comprehension with an unsatisfiable predicate[5] to construct a set with no elements, the empty set. To the symbol 0, we correspond the empty set. Next we recursively define $S(w) = w \cup \{w\}$. If w is an ordinal, then we say $S(w)$ is also an ordinal. We construct the von Neumann ordinals in this way. The first few can be given as follows.

- $0 : \emptyset$
- $1 : \{\emptyset\}$
- $2 : \{\emptyset, \{\emptyset\}\}$
- $3 : \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
- $4 : \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

[5]Such a predicate could be $\Phi(x) = (x \land \neg x)$

A set theoretic definition of a number is that which is a set of all sets representing numbers strictly less than it. The ordinal for 2 is a set containing the ordinals for 0,1. This allows us to easily define the relation $x < y$ on ordinals as the relation $x \in y$ as sets. We may now apply the axiom of existentionality to prove two numbers are equal by showing the sets that represent them are. Is it true that $\forall z[z \in \{\emptyset\} \iff z \in \{\emptyset\}]$? Yes, we then conclude that $1 = 1$. Its worth nothing there are other ways to construct the natural numbers from set theory.

5 Russell's Paradox

Bertrand Russell noticed the following issue which applied Begriffsschrift and many axiomatic systems. Recall our said axioms:

- Existentionality: $\forall x \forall y[\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$
- Unrestricted Comprehension: $\exists y \forall x[x \in y \Leftrightarrow \Phi(x)]$

Suppose we had the predicate $\Phi(x) = x \notin x$. A perfectly valid predicate. It defines sets which do not contain themselves. By the axiom of unrestricted comprehension, there exists a set containing all sets which do not contain themselves.

$$\exists y \forall x[x \in y \Leftrightarrow x \notin x]$$

Since it's true $\forall x$, we may perform specification, and consider the case for one x. What happens for $x = y$?

$$y \in y \iff y \notin y$$

A contradiction! Wait! No! We are not in a proof by contradiction, yet we have derived an inconsistency where a statement is true and false simultaneously. We have shown that Frege's axiomatic system was capable of producing an inconsistency, and is therefore, inconsistent.

Two remarks on this proof. First, is that the name of this proof technique is diagonalization. It has a similar structure to Cantor's proof of the existence of uncountable sets. You use an argument of negated self-reference to construct a useless object whose only job is to induce contradiction. A second note, is this is really a formalization of a classic semantic paradox: "This sentence is false".

6 Hilbert's Program

There are, were, many competing schools of thought among the thinkers in the philosophy of mathematics. Constructivists, finitists, intuitionists, logicists, formalists, linguicists, and more. We focus on the formalists and logicists. They seek a secure, rigorous, and logical foundation in which to secure all of mathematics. Roughly with these goals:

1. All mathematics written in a precise formal language manipulated according to well-defined rules, where objects in the system are independent of meaning.

2. Completeness: $\forall p$, there exists a proof of p if it's true, or a proof of $\neg p$ if it's false. This asserts provable $\iff$ true. A system being complete means in some sense, it is "total". From the axioms, all statements are provable. There is no theorem which requires some missing secret axiom. It also asserts if something is true, there must exist a proof of it, and a way to deduce such a proof.

3. Consistency: $\forall p$ there exists a proof of $p \wedge \neg p$. Every statement is exactly true or exactly false. No statement can be false and true simultaneously. A system being consistent is the bare minimum requirement for it being useful. This asserts you cannot prove $0 = 1$. If you could, then everything follows trivially.

4. Decidability: There should exist an algorithm to decide the truth value of any statement.

In Formulario Mathematico, Peano develops a symbolic language for mathematics. He says

> Each professor will be able to adopt this Formulario as a textbook, for it ought to contain all theorems and all methods. His teaching will be reduced to showing how to read the formulas, and to indicating to the students the theorems that he wishes to explain in his course.

This was a massive effort by many people to rigorously formalize and mechanize all fields of mathematics.

7 Avoiding Paradox

In order to carry out Hilbert's program, it was necessary to eliminate paradox. Natural language is full of ambiguity, and in this formalization, the semantic paradoxes we speak came under greater scrutiny. Many paradoxes revealed themselves to not be inconsistencies as all, such as Galileo's paradox, or Zeno's paradox of Achillies and the tortise. Galileo's paradox asserted that the set of numbers could not be a set since it violated the Aristotelian principle. Cantor resolved this by taking the fact that the property does not apply to infinite sets. Some paradoxes did not resolve themselves:

- Russell's Paradox: Does the set of all sets contain itself?
- Liar's Paradox: If someone says "I am lying", is the statement true or false? If it is true, then they are lying, so it is false. If it is false, then they are not lying, so it must be true.
- "Cretans are always liars" - Epimendes of Crete.
- Cervantes in Don Quixote (1605): A traveller has fallen among cannibals. They offer him the opportunity to assert a sentence. If it is true, he is boiled. If it is false, he is roasted. What should he say?[6]

[6] I will be roasted

In these cases, the problem appears to be self-reference. The obvious conclusion is that a formal system of mathematics must prevent the ability to construct self-referential statements. The first change in this regard was to replace the axiom of unrestricted comprehension, with a restricted variant.

- unrestricted comprehension:
$$\{x \mid \Phi(x)\}$$

- restricted comprehension:
$$\{x \subseteq z \mid \Phi(x, z)\}$$

Now you can only construct subsets of other sets. By restricting comprehension, we appear to avoid Russell's paradox. The set of all sets is not a set, and cannot be constructed.

The most significant effort to lay a foundation for all of mathematics is owed to Russell and Whitehead. They spent decades and thousands of pages to build up Principia Mathematica, an axiomitization along with a "Theory of Types." To give a quick summary of twenty years of work, they hoped to avoid self-reference by using these types. Anything of some type i is unable to construct sentences which reference other things of type i (including itself). There exists a hierarchy of objects of different types. Principia Mathematica was thousands of pages, three volumes worth and their lifes work. They gave axiom after axiom and theorem after theorem. It supposedly took them 172 pages to prove $1+1=2$ following their system. After Principia Mathematica was written, there was much effort to prove it was both consistent and complete. That is to show $PM \nvdash (0=1)$, and anything that was true had a proof within the axioms of PM. We will discuss if this effort suceeded next time.

8 Logicomix

`https://archive.org/details/Logicomix-Comic-EarlyLifeOfBertrandRussell`