CS 4510 Automata and Complexity February 5th 2023

Lecture 7: Pushdown Automata

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1 Introduction

We mentioned previously how if we had a stack data structure, we could parse arithmetical expressions, like a classic data structures assignment. Lets do that. We are literally going to give an NFA a stack. We say a Pushdown Automata (PDA) is a tuple $(Q, \Sigma, \Gamma, q_0, \delta, F)$

- $Q = \{q_0, \ldots, q_k\}$ our finite set of states
- Σ = finite input alphabet
- Γ = finite stack alphabet. By convention, we will usually set $\Gamma = \Sigma \cup \{\$\}$, where \$ is a special symbol we denote as the stack canary.
- $q_0 \in Q$ is the denoted start state
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\}) = \mathcal{P}(Q \times (\Gamma \cup \{\varepsilon\}))$ our transition function.
- $F \subseteq Q$ is the set of final or accepting states.

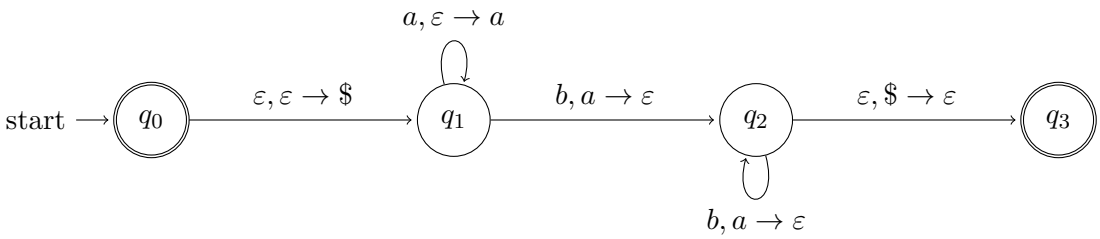
Notice that we have basically done as we said we would do. We just augmented an NFA to give it a stack. The transition function works as follows. You go from some state, optionally read a symbol off the input, and optionally pop the top of the stack. Then you transition to a new state and optionally push something onto the stack. When you see a transition of the form $a, b \to c$, you read a from input, pop b and push c onto the stack. The transition may only be taken if b is the symbol at the top of the stack and a is the next symbol to read in the input. Note that our defined PDA is explicitly nondeterministic. Deterministic PDAs exist, but unlike DFAs and NFAs, they are explicitly weaker. We will not cover them. Lets give some programming analogies to the types of transitions possible.

- $\varepsilon, \varepsilon \to \varepsilon$ We read nothing from the input, pop nothing, and push nothing. This means we really only change states, so this acts like an ε-transition in an NFA.
- $a, \varepsilon \to a; b, \varepsilon \to b$ We read a, b off the input, popped nothing, and pushed a, b respectively. We would only push exactly what we read, so this would push the input to the stack. If it was in a self-loop, it would push the entire input to the stack.
- $\varepsilon, a \to \varepsilon; \varepsilon, b \to \varepsilon$ Read nothing from the input, pop something, and push nothing. This would pop the top of the stack, whatever it may be. In a self-loop, it would dump the stack.
- $a, b \to b$ Read a off the input, pop b off the stack, then push b right back. This allows us to essentially transition conditionally off of "peeking" the top of the stack. We cannot read the top of the stack without popping it, but we can simulate this as popping it and then pushing it right back.

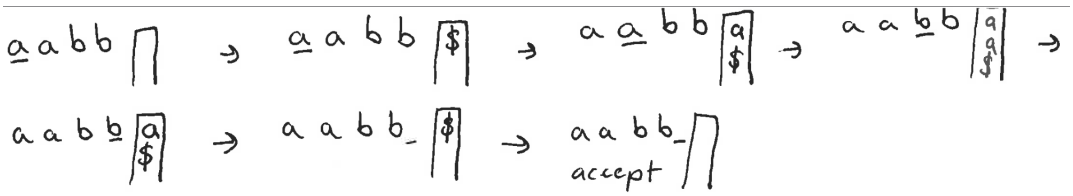
2 Examples

2.1 $\{a^n b^n \mid n \in \mathbb{N}\}$

While reading a's off the input, we are going to push them. If we read the symbol b, we start reading b's off the input, matching them to a's popping off the stack, and we accept only if we read as many b's as we previously pushed a's. We don't have an inbuilt mechanism to determine if the stack is empty. Thats why we begin almost every PDA by pushing \$. Then if we ever see our canary again, we know the stack is empty.



Note that we must also accept ε so the start state is marked as accepting. Consider the behavior of this PDA on input *aabb*. There may be many rejecting computations, but we will show one accepting one. We push the a's onto the stack, and upon reading b's we pop an a for each b, matching them together. We accept only when both the stack and input are empty to ensure the number of a's is equal to the number of b's.

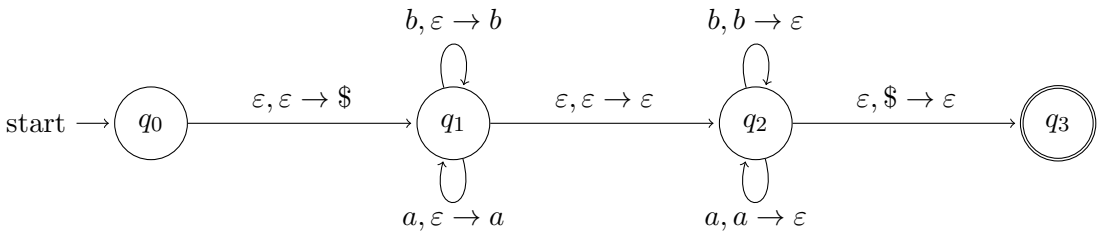


Note here as we compute, we have a sequence of configurations of the machine. A configuration is an encoding of all pieces of a machine at some current computation step. For our PDA, that means the current state[1], how much has been read from the input, and the contents and order of the stack. Think of it like an instantaneous snapshot. The reason we do not really care about the configuration of a DFA or NFA is that its simply just the current state and how much input has been read. Lets consider the behavior of the machine on input $a^{n+1}b^n$. It will push the $n+1$ a's, match them to b's. At state q_2, there will be one a left on the top of the stack with nothing left in the input. With no transitions beginning with ε, ε or ε, a, we are forced to implicitly reject from q_2. Consider the behavior of the machine on input $a^n b^{n+1}$. We will push the n a's onto the stack, then pop them all off matching to the b's. The stack will only contain the canary with one b still left to read in the input. The PDA must read the entire input, so we are forced to implicitly reject. If there is any string which has a's and b's out of order, we also implicitly reject from q_3.

[1] Even though this sequence of configurations does not list the curent state.

2.2 $\{ww^R \mid w \in \Sigma^*\}$

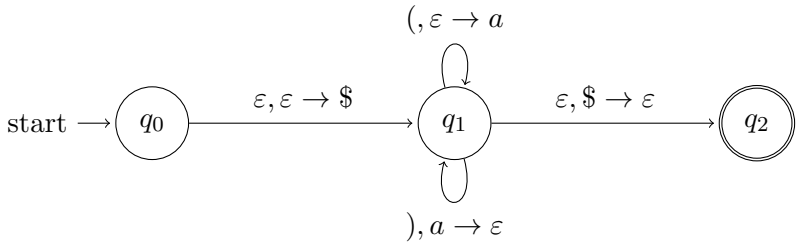
As you might suspect, our PDA will be similar, however before we could start matching the second half of our string by seeing a b. Here, we don't have that privilege. We can still solve it by nondeterministically guessing the midpoint of the input!



Note that all computation paths which incorrectly guess the midpoint of the string would reject, implicitly or otherwise. We also need not mark the start state as accepting as there exists a non-trivial computation path to accept ε. We would push the canary, epsilon transition, pop the canary, and accept.

2.3 Dyck Language

Recall that the Dyck language is the set of balanced parenthesis. You decide this language trivially with a counter. You increment your counter for every open you see, and decrement it for every close. If your counter ever goes negative, you reject. If your counter ends with anything other than zero, you reject. We will use the stack more simply here, as just a unary counter.



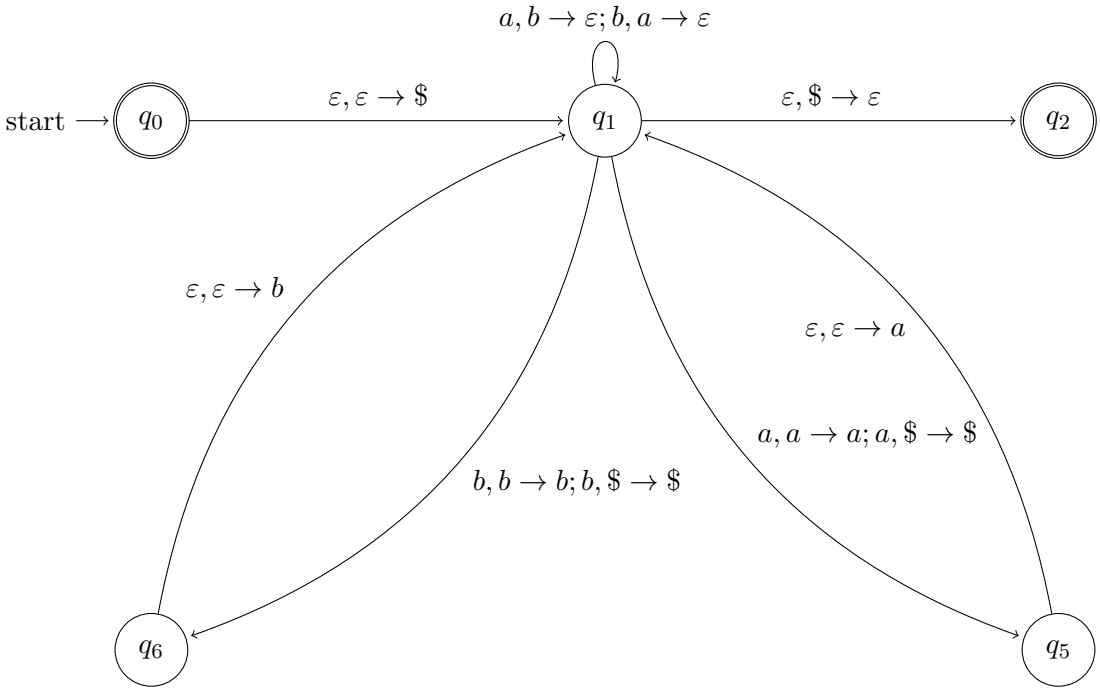
Here $\Sigma = \{(,)\}$ and $\Gamma = \{a, \$\}$. Just for clarity, I chose Γ to be something else, as it doesn't particularly matter what we count with.

You may also note the difference between programming NFAs vs PDAs. Our PDAs usually have the convention of beginning with pushing the canary, and end with popping it. We usually want to accept when the stack is empty but this is not a requirement for all PDAs. Second, notice how straight-line our PDAs look compared to NFAs, which may have many transitions going everywhere. Think of the states of our PDA like sequential lines of code from the start to the end.

2.4 $\{w \in \Sigma^* \mid \#a(w) = \#b(w)\}$

This will be similar to two PDAs we have seen, the Dyck language, and $a^n b^n$. However, instead of a positive counter, we would need a counter which could occasionally be negative.

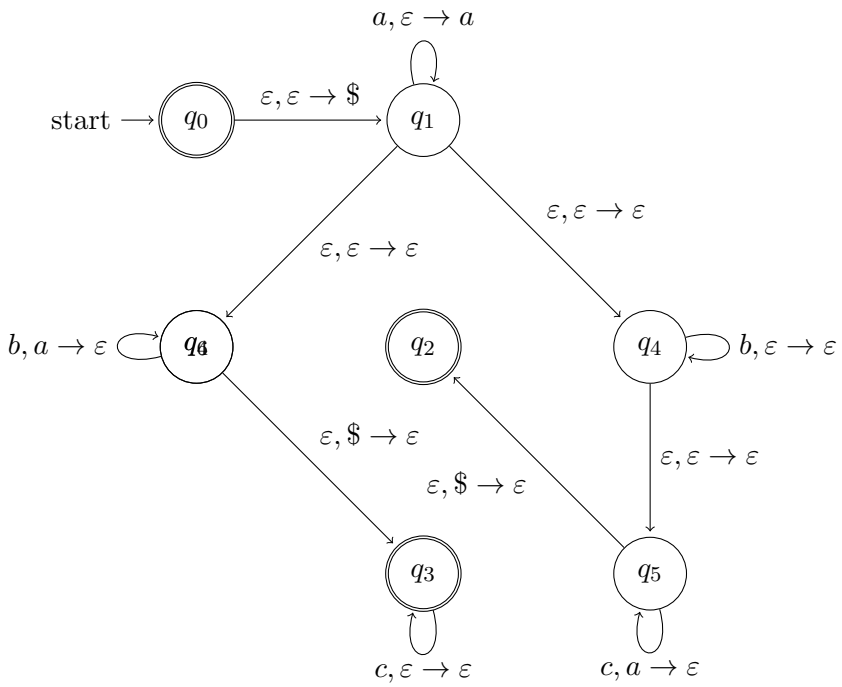
Different symbols on the stack will be used to represent the surplus in one direction or the other.



You should note we could do this PDA far simpler if we were allowed to push more than one symbol onto the stack per transition. We will show how to do this next lecture.

2.5 $\{a^i b^j c^k \mid i, j, k \in \mathbb{N}, i = j \textbf{ or } j = k\}$

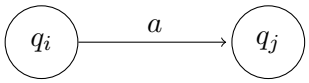
Since the definition of the language contains an or, why don't we make two PDAs for each case, and then use nondeterminism to join them together.

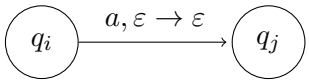


Although we won't show it, this PDA requires nondeterminism. There is no DPDA to decide this language.

3 $\mathscr{L}(NFA) \subsetneq \mathscr{L}(PDA)$

We can convert any NFA into a PDA by simply preserving the topology of the states, and ignoring the stack. For some $q_j \in \delta(q_i, a)$ in our NFA, we would have $(q_j, \varepsilon) \in \delta(q_i, a, \varepsilon)$ in our PDA.





Our containment is strict because we gave PDAs for many languages we know not to be regular. In fact, all the examples we gave were languages we know to be context free. Next time we will show that $\mathscr{L}(PDA) = \mathscr{L}(CFG)$.