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Abstract

In this paper, we present a game-theoretic analysis of a visibility-based pursuit–evasion game in a planar environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. Both players have a complete map of the environment. Both players have omnidirectional vision and have knowledge about each other's current position as long as they are visible to each other. The pursuer wants to maintain visibility of the evader for the maximum possible time and the evader wants to escape the pursuer's sight as soon as possible. Under this information structure, we present necessary and sufficient conditions for surveillance and escape. We present strategies for the players that are in Nash equilibrium. The strategies are a function of the value of the game. Using these strategies, we construct a value function by integrating the adjoint equations backward in time from the termination situations provided by the corners in the environment. From these value functions we recompute the control strategies for the players to obtain optimal trajectories for the players near the termination situation. This is the first work that presents the necessary and sufficient conditions for tracking for a visibility based pursuit–evasion game and presents the equilibrium strategies for the players.

KEY WORDS—game theory, pursuit-evasion

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Figures 4–8 appear in color online: http://ijr.sagepub.com

1. Introduction

Consider a situation in which a group of mobile pursuers having bounded speeds are trying to keep sight of an unpredictable evader in a cluttered environment. In order to deploy minimum number of pursuers needed to track the evader it would be useful to know the best strategy that can be used by a single pursuer. In this work, we analyze the problem of a mobile pursuer trying to track a mobile evader in an environment containing obstacles. Both the pursuer and the evader are holonomic with bounded speeds and can see each other at the beginning of the game. The players do not have knowledge of each other's future actions. We formulate the problem of tracking as a game in which the goal of the pursuer is to keep the evader in its field of view for maximum possible time and the goal of the evader is to escape the pursuer's field of view in minimum time by breaking the line of sight around a corner.

An interesting application of this problem is in security and surveillance systems. It may be useful for a security robot to track a malicious evader that is trying to escape. Also, an 'evader' may not be intentionally trying to slip out of view. A pursuer robot may simply be asked to continuously follow and monitor at a distance an evader performing a task not necessarily related to the target tracking game (Becker 1995). The pursuer may also be monitoring the evader for quality control, verifying the evader does not perform some undesired behavior, or ensuring that the evader is not in distress. The results are useful as an analysis of when escape is possible. If it is impossible to slip away, it may be desirable for the evader to immediately surrender or undertake a strategy not involving escape. In home care settings, a tracking robot can follow elderly people and alert caregivers of emergencies. Target-tracking techniques in the presence of obstacles have been proposed for the graphic animation of digital actors, in order to select the successive viewpoints under which an actor is to be displayed as it moves in its environment (Li et al. 1997).

In the past, we have addressed tracking problems similar to the one described in this paper. In Bhattacharya et al. (2007), we addressed the problem of a pursuer trying to track an antagonistic evader around a single corner. We partition the visibility region of the pursuer into regions based on the strategies used by the players to achieve their goals. Based on these partitions we propose a sufficient condition of escape for the evader in general environments. In Bhattacharya and Hutchinson (2008a), given the initial position of the evader in a general environment, we use the sufficient condition to compute an approximate bound on the initial positions of the pursuer from which it might track the evader. In this work, we formulate the target-tracking problem as a game in which the pursuer wants to maximize the time for which it can track the evader and the evader wants to minimize it. We compute the strategies for the players that are in *Nash equilibrium*. If a player follows its equilibrium strategy, it is guaranteed of a minimum outcome without any knowledge of the other player's future actions. Moreover, when a pair of strategies for the players is in *Nash equilibrium* then any unilateral deviation of a player from its equilibrium strategy might lead to a lower outcome for it. Consider a situation in which the pursuer can keep the evader in sight for time t_f when the players follow their equilibrium strategies. If the evader deviates from its equilibrium strategy, then the pursuer has a strategy to track it for a time greater than t_f. On the other hand, if the pursuer deviates from its equilibrium strategy, then the evader can escape in time less than t_f. Hence, there is no motivation for either of the players to deviate from their equilibrium strategies due to the lack of knowledge of the other player's future actions. For a pair of equilibrium strategies for the players either the evader can escape the pursuer's sight in finite time or the pursuer can track the evader forever. Hence, computing them gives us the strategies sufficient for tracking or escape, whichever holds at a given point in the state space. This is the first work that addresses the necessary and sufficient conditions for tracking and provides equilibrium strategies for the players. We use these strategies to integrate the kinematic equations of the system backward in time from the termination situations to obtain the optimal trajectories for the players.

Prior work regarding the problem of tracking is based on discretizing the motion models of the players or the state space in which the game is being played (LaValle et al. 1997; Bandyopadhyay et al. 2006). These techniques lead to approximate numerical solutions that become computationally inefficient with increasing time horizon of the game. Moreover, they assume a prior model of uncertainty for the evader's future actions. In contrast to these works, we use continuous-time motion models for the players and provide closed-form solutions to the coupled non-linear differential equations that govern our system kinematics. Hence, no error is introduced in the solutions due to discretizations of any form. Further, our results are valid for scenarios in which the players have no knowledge about each others future actions.

In this work, we use differential games to analyze a pursuit–evasion problem. The theory of deterministic pursuit–evasion was single-handedly created by Isaacs that culminated in his 1965 book (Isaacs 1965). An exhaustive analysis of solved and partly solved zero-sum differential games is provided in Başar and Olsder (1999) and Lewin (1994). Most of the classical problems in pursuit–evasion deal with players in obstacle-free space having either constraints on their motion or constraints on their control due to under-actuation. In the recent past, researchers have analyzed pursuit–evasion problems with constraints in the state space. In Melikyan and Hovakimyan (1991a,b, 1993), a pursuit–evasion game is analyzed with the pursuer and the evader constrained to move on a two-dimensional conical surface in a three-dimensional space. Our work belongs to this category of problems. In our problem, the state constraints arise due to the presence of obstacles that obstruct visibility as well as motion of the players in the workspace and the control constraints arise as a result of the bounded speed of the players. Apart from these problems, researchers have also analyzed pursuit–evasion in $\mathbb{R}^n$ (Kopparty and Ravishankar 2005), in non-convex domains of arbitrary dimension (Alexander et al. 2006), in unbounded domains (Alexander et al. 2008) and in graphs (Parsons 1976).

In Section 2, we present the formulation of the game. In Section 3, we analyze the termination situations presented by the obstacles around any corner in the environment. In Section 4, we present the strategies for the players that are in *Nash equilibrium*. In Section 5 we present the construction of the optimal trajectories. In Section 6, we present the conclusions and the future work.

2. Formulation of the game

We consider a mobile pursuer and an evader moving in a plane with velocities $u = (u_p, \theta_p)$ and $v = (u_e, \theta_e)$, respectively. Here u_p and u_e are the speeds of the players that are bounded by $\overline{v}_p$ and $\overline{v}_e$, respectively, and θ_p and θ_e are the direction of the velocity vectors. We use r to denote the ratio of the maximum speed of the evader to that of the pursuer $r = \overline{v}_e/\overline{v}_p$. They are point robots with no constraints in their motion except for bounded speeds. The workspace contains obstacles that restrict pursuer and evader motions and may occlude the pursuer's line of sight to the evader. The initial position of the pursuer and the evader is such that they are visible to each other. The visibility region of the pursuer is the set of points from which a line segment from the pursuer to that point does not intersect the obstacle region. Visibility extends uniformly in all directions and is only terminated by workspace obstacles (omnidirectional, unbounded visibility). The pursuer and the evader know each others current position as long as they can see each other. Both players have a complete map

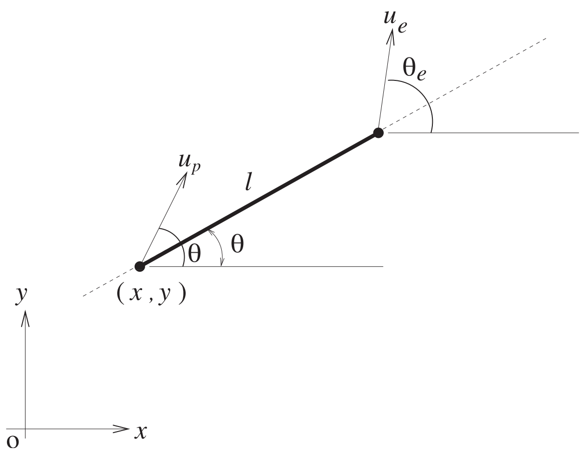

Fig. 1. State variables and control inputs.

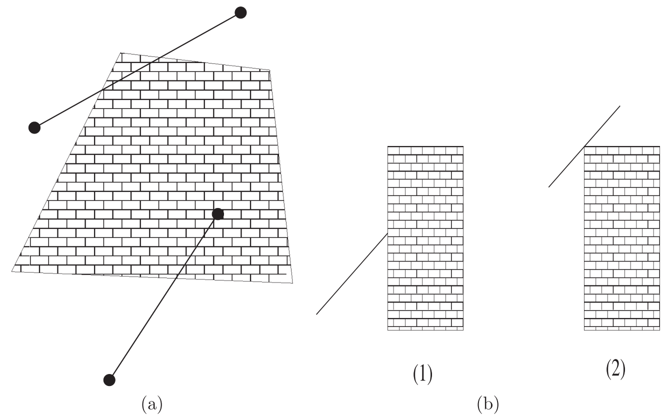

Fig. 2. (a) Workspace obstacles and (b) types of contacts on the boundary.

of the environment. In this setting, we consider the following game. The pursuer wants to keep the evader in its visibility region for the maximum possible time and the evader wants to break the line of sight to the pursuer as soon as possible. If, at any instant, the evader breaks the line of sight to the pursuer, the game terminates. Given the initial position of the pursuer and the evader, we want to know the optimal strategies used by the players to achieve their respective goals. Optimality refers to the strategies used by the players that are in *Nash equilibrium*.

We model the system as a non-rigid bar of variable length representing the line of sight between the pursuer and the evader. The bounded velocities of the pursuer and the evader are modeled as control inputs at opposite ends of the bar. Any occlusion between the pursuer and the evader leads to a situation in which the bar intersects the obstacles. Hence, the pursuer's goal is to keep the bar in free space for the maximum possible time and the evader's goal is to force the bar to intersect some obstacle as soon as possible. In this work we assume that the line of sight is not blocked due to grazing contact with the boundary. Hence, visibility is retained even if a vertex in the environment is incident on the bar.

Figure 1 shows the configuration of the system along with the state variables and the control inputs. Here (x, y) is the position of the end of the bar controlled by the pursuer, l is the length of the bar and θ is the angle made by the bar with the horizontal line. The configuration of the system can be expressed as (x, y, l, θ) and hence it is $\mathbb{R}^3 \times S^1$. In the rest of the paper, $\mathbf{x}$ $(\in \mathbb{R}^3 \times S^1)$ will be used to represent the state of the bar. The pursuer controls the velocity, u, of one end of the bar and the evader controls the velocity, v, of the other end of the bar. The differential equation describing the kinematics of the system is given by the following equation:

$$\begin{pmatrix} \dot{x} \\ y \\ l \\ \theta \end{pmatrix} = \begin{pmatrix} u_p \cos\theta_p \\ u_p \sin\theta_p \\ u_e \cos(\theta_e - \theta) - u_p \cos(\theta_p - \theta) \\ \frac{u_e}{l}\sin(\theta_e - \theta) - \frac{u_p}{l}\sin(\theta_p - \theta) \end{pmatrix}.$$

The above equation can also be expressed in the form $\dot{\mathbf{x}} = f(\mathbf{x}, u, v)$.

3. State Constraints and Termination Situations

In this section, we present a description of the obstacles in the configuration space. The workspace contains polygonal obstacles in the plane that obstruct the visibility and motion of the players. Since the system is modeled as a bar representing the line of sight between the players, the obstruction of mutual visibility as well as the motion of the players caused due to obstacles in the workspace can be expressed as a state constraint in $\mathbb{R}^3 \times S^1$. These state constraints can be expressed as configuration space obstacles. In $\mathbb{R}^3 \times S^1$, the configuration space obstacles are the set of all (x, y, l, θ) such that the bar has a non-empty intersection with some obstacle in the workspace. Figure 2 shows two such configurations of the bar that lies in configuration space obstacles. In one configuration the obstacle blocks the line of sight between the pursuer and the evader. In the other configuration a player is inside the obstacle which is forbidden according to the rules of the game.

The *game set* is the set of all points in $\mathbb{R}^3 \times S^1$ that belong to the free space. Hence, the boundary of the game set is the same as the boundary of the configuration space obstacles. The boundary of the game set consists of two kinds of contact between the bar and the obstacles. Refer to Figure 2(b). The first kind of contact occurs when at least one end of the bar touches an obstacle in the plane. At no point in time can the state of the game cross the boundary at such a point, as this is equivalent to either of the players penetrating into an obstacle in the workspace. The second kind of contact occurs when a vertex of an obstacle is incident on the bar and this set of points on the boundary of the game set is called the *Target set*. At any point in time, if the current state of the game lies on the target set, then it can cross the boundary according to the rules of the game since in the workspace this is equivalent to breaking the mutual visibility between the players which results in the termination of the game. Since we are interested in situations where the mutual visibility between the players can be broken, we are only interested in the part of the boundary that forms the target set.

In this game, termination occurs only when the evader can break the line of sight to the pursuer around a corner. Every corner in the environment presents an opportunity for the evader to break the line of sight. Hence, every corner presents a termination situation for the game.

If the state of the system lies on the target set, then a vertex of some obstacle is incident on the bar. The evader cannot guarantee termination at every point on the target set. Figure 3 shows a configuration of the bar in which the system is on the target set. Let d_p denote the distance of the vertex from (x, y) which is same as the distance of the pursuer from the vertex. Let l denote the length of the bar which is same as the distance between the pursuer and the evader. The evader can force termination if and only if the maximum angular velocity of the evader around the corner is greater than the maximum angular velocity achievable by the pursuer around the corner. This can happen if and only if

$$\frac{d_p}{l} > \frac{1}{1+r}.$$

Hence, we can further subdivide the target set depending on whether the evader can guarantee termination at that point. The part of the target set where evader can guarantee termination regardless of the choice of the controls of the pursuer is called the *usable part* (UP). The remaining part of the target set outside the UP is called the *non-usable part* (NUP) and the game will never terminate on the NUP. Given any initial position of the pursuer and the evader, the game will always terminate on the UP.

We now present the equations characterizing the target set around a vertex of an obstacle. Refer to Figure 3. The figure shows a configuration of the bar in which a vertex, v, lies on the bar. Hence, the current state of the bar lies on the target set. We want the equation of the hyperplane that characterizes the target set generated by v. Let (x, y, l, θ) be the configuration of the bar and (x^o, y^o) be the coordinates of the vertex of the obstacle. Let $\lambda \in (0, 1)$ be a variable that determines the fraction of the length of the bar between (x, y) and the corner (x^o, y^o). We can write the following equations of constraints for the bar.

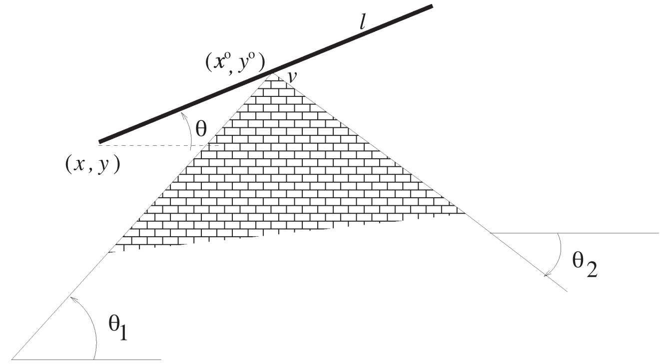

Fig. 3. A configuration of the bar on the target set.

$$x^o - x = \lambda l \cos\theta,$$

$$y^o - y = \lambda l \sin\theta.$$

In the above equation, as λ changes, the point of contact between the bar and the vertex changes. Hence, the target set is characterized by the following equation:

$$\Rightarrow F(x, y, l, \theta) = (y^o - y)\cos\theta - (x^o - x)\sin\theta = 0. \quad (1)$$

Since the above equation applies to any $\lambda \in (r/(1+r), 1)$, Equation (1) also characterizes the usable part of the target set.

Given a vertex, the target set generated by it in the configuration space has the following boundaries.

- The pursuer lies on the corner $\Rightarrow (x, y) = (x^o, y^o)$.
- The evader lies on the corner $\Rightarrow (x + l\cos\theta, y + l\sin\theta) = (x^o, y^o)$.
- The bar is parallel to either of the edges incident on the vertex: $\theta = \theta_2$ or $\theta = \theta_1$.

Every vertex will generate a target set. The final boundary of the target set generated by a vertex will depend on the position of the other vertices and edges in the environment. However, the equation of the target set will be given by Equation (1).

The unit normal to a point (x, y, l, θ) on the target set is given by

$$n(x, y, l, \theta) = \nabla F = \frac{1}{\sqrt{1 + (x^o - x)^2 \sec^2\theta}} \times [\sin\theta \quad -\cos\theta \quad 0 \quad -(x^o - x)\sec\theta]^{\mathrm{T}}. \quad (2)$$

4. Optimal Strategies

In this section we present the optimal controls for the players. Before we define the concept of optimality we need to define the payoff for the players in the game. Consider a game that terminates at time t_f. Since the pursuer wants to increase the time of termination its payoff function can be considered as t_f. On the other hand, since the evader wants to minimize the time of termination its payoff can be considered to be $-t_f$. Since the payoff functions of the players add to zero, this is a *zero sum* game. Another way to show that it is a zero sum game is to observe that the pursuer's gain is equal to the evader's loss and vice versa. The time of termination is a function of the initial state $\mathbf{x_0}$ and the control history during the play, u and v. Let π denote the functional $\pi : (\mathbf{x_0}, u, v) \rightarrow t_f \in \mathbb{R}$. Here π is called the *outcome functional* and is given by the following expression:

$$\pi[\mathbf{x}_0, u, v] = \int_0^{t_f} L[\mathbf{x}(\tau), u(\tau), v(\tau)]\, d\tau + G[\mathbf{x}(t_f)],$$

In the above expression $L[\mathbf{x}(\tau), u(\tau), v(\tau)]$ is called the *running cost function* and $G[\mathbf{x}(t_f)]$ is called the *terminal cost function*. The running cost function is the cost incurred while the game is being played. The terminal cost function is the cost incurred for reaching a particular terminal state on the target set. In this game, $L[\mathbf{x}(\tau), u(\tau), v(\tau)] = 1$ and $G[\mathbf{x}(t_f)] = 0$. The pursuer wants to maximize the outcome functional and the evader wants to minimize it.

For a point $\mathbf{x}$ in the state space, $J(\mathbf{x})$ represents the outcome if the players implement their optimal strategy starting at the point $\mathbf{x}$. It is the time of termination of the game when the players implement their optimal strategies. It is also called the *value* of the game at $\mathbf{x}$. Any unilateral deviation from the optimal strategy by a player can lead to a better payoff for the other player. For example, for a game that starts at a point $\mathbf{x}$, if the evader deviates from the optimal strategy then there is a strategy for the pursuer in which its payoff is greater than $J(\mathbf{x})$ and if the pursuer deviates from the optimal strategy then there is a strategy for evader in which its payoff is greater than $-J(\mathbf{x})$. Since this is a *zero sum* game, any strategy that leads to a higher payoff for one player will reduce the payoff for the second player.

Here $\nabla J = [J_x \quad J_y \quad J_l \quad J_\theta]^{\mathrm{T}}$ denotes the gradient of the value function. The Hamiltonian of such systems is given by the following expression:

$$H(\mathbf{x}, \nabla J, u, v) = \nabla J \cdot f(\mathbf{x}, u, v) + L(\mathbf{x}, u, v)$$

Let $u^* = (u_p^*, \theta_p^*)$ and $v^* = (u_e^*, \theta_e^*)$ be the optimal controls used by the pursuer and the evader respectively. From the definition of the value of the game we can conclude that $J(\mathbf{x_0}) = \pi[\mathbf{x}_0, u^*, v^*]$. The Hamiltonian of the system satisfies the following conditions along the optimal trajectories (Isaacs 1965). These are called the *Isaacs* conditions:

1. $H(\mathbf{x}, \nabla J, u, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v)$;
2. $H(\mathbf{x}, \nabla J, u^*, v^*) = 0$.

Condition 1 implies that when the players implement their optimal strategies any unilateral deviation by the pursuer leads to a smaller value for the Hamiltonian and any unilateral deviation by the evader leads to a larger value of the Hamiltonian. Moreover, condition 2 implies that when the players implement their optimal controls the Hamiltonian of the system is zero. The *Isaacs* conditions are an extension of the *Pontryagin's principle* in optimization to a game.

The Hamiltonian of our system is given by the following expression:

$$\begin{aligned} H(\mathbf{x}, \nabla J, u, v) &= \nabla J \cdot f(\mathbf{x}, u, v) + L \\ &= u_p \Big[J_x \cos\theta_p - J_l \cos(\theta_p - \theta) \\ &\quad - \frac{J_\theta}{l} \sin(\theta_p - \theta) + J_y \sin\theta_p \Big] \\ &\quad + u_e \left[J_l \cos(\theta_e - \theta) + \frac{J_\theta}{l} \sin(\theta_e - \theta) \right] + 1. \end{aligned}$$

Since the evader wants to minimize the time of escape and the pursuer wants to maximize the time of escape, Isaacs first condition requires the following to be true along the optimal trajectories:

$$(u_e^*, \theta_e^*, u_p^*, \theta_p^*) = \min_{u_e, \theta_e} \max_{u_p, \theta_p} H(x, \nabla J, u, v). \quad (3)$$

We can see that the Hamiltonian is *separable* in the controls u_p and u_e, i.e. it can be written in the form $u_p f_1(\mathbf{x}, \nabla J) + u_e f_2(\mathbf{x}, \nabla J)$. Hence, the optimal controls for the players are given by the following expressions in terms of the gradient of the value function:

$$\begin{aligned} (\cos\theta_p^*, \sin\theta_p^*) &\Big\| \Big(J_x - J_l \cos\theta + \frac{J_\theta}{l} \sin\theta, J_y \\ &\quad - J_l \sin\theta - \frac{J_\theta}{l} \cos\theta \Big), \\ (\cos(\theta_e^* - \theta), \sin(\theta_e^* - \theta)) &\Big\| \left(-J_l, -\frac{J_\theta}{l} \right), \\ u_e^* &= \overline{v}_e, \\ u_p^* &= \overline{v}_p. \end{aligned} \quad (4)$$

Owing to the lack of space, the derivation is presented elaborately in Bhattacharya and Hutchinson (2008b). In the first and the second equation $\|$ is used to denote parallel vectors. In case

$$J_x - J_l \cos\theta + \frac{J_\theta}{l} \sin\theta = 0$$

and

$$J_y - J_l \sin\theta - \frac{J_\theta}{l} \cos\theta = 0,$$

then θ_p^* can take any value and the pursuer can follow any control strategy. Similarly if $J_l = 0$ and $J_\theta/l = 0$, then θ_e^* can take any value and the evader can follow any control strategy. These conditions represent *singularity* in the Hamiltonian.

The entire game set can be partitioned into two regions depending on the value of the game. For all the initial positions of the pursuer and the evader for which the value of the game $J(\mathbf{x})$ is finite, the evader can break the line of sight in finite time by following the strategies in Equation (4). For all of the initial positions of the pursuer and the evader for which the value of the game is infinite, the pursuer can track the evader forever if it follows the controls given in Equation (4). Hence, Equation (3) are the necessary and sufficient conditions for pursuer to track the evader in terms of the Hamiltonian of the system.

The analysis performed in this section implies that if we are given the value function $J(\mathbf{x})$, then we can compute the optimal strategies for the players by using Equation (4).

5. Construction of Optimal Trajectories

In this section we present the trajectories generated by the optimal control laws that terminate on the UP. We use the following theorem to construct the optimal trajectories.

Theorem 1 (Isaacs (1965)). *Along the optimal trajectory, the following equation holds:*

$$\frac{d}{dt}\nabla J[\mathbf{x}(t)] = -\frac{\partial}{\partial x}H(\mathbf{x}, \nabla J, u^*, \upsilon^*).$$

The above equation is called the adjoint equation *and the components of* $\nabla J(\mathbf{x})$ *are called adjoint variables. The retro-time (*time-to-go*) form of the adjoint equations is*

$$\frac{d}{d\tau}\nabla J[\mathbf{x}(\tau)] = \frac{\partial}{\partial x}H(\mathbf{x}, \nabla J, u^*, \upsilon^*),$$

where $\tau = t_f - t$ *is called the retro-time and* t_f *is the time of termination of the game.*

The adjoint equation is a differential equation for the gradient of the value function $J(\mathbf{x})$ along the optimal trajectories in terms of the optimal controls. Since Equation (4) gives the optimal controls of the players as a function of $\nabla J(\mathbf{x})$, we integrate the adjoint equations backward in time from the UP to obtain $\nabla J(\mathbf{x})$ in terms of the state variables. Substituting $\nabla J(\mathbf{x})$ into the optimal controls gives a feedback control strategy for the players. Substituting the feedback control laws for the players into the kinematic equation leads to the optimal trajectories. Owing to a lack of space, the construction of the optimal trajectories is provided elaborately in Bhattacharya and Hutchinson (2008b).

From the analysis performed in Bhattacharya and Hutchinson (2008b), we present the optimal trajectories of the players.

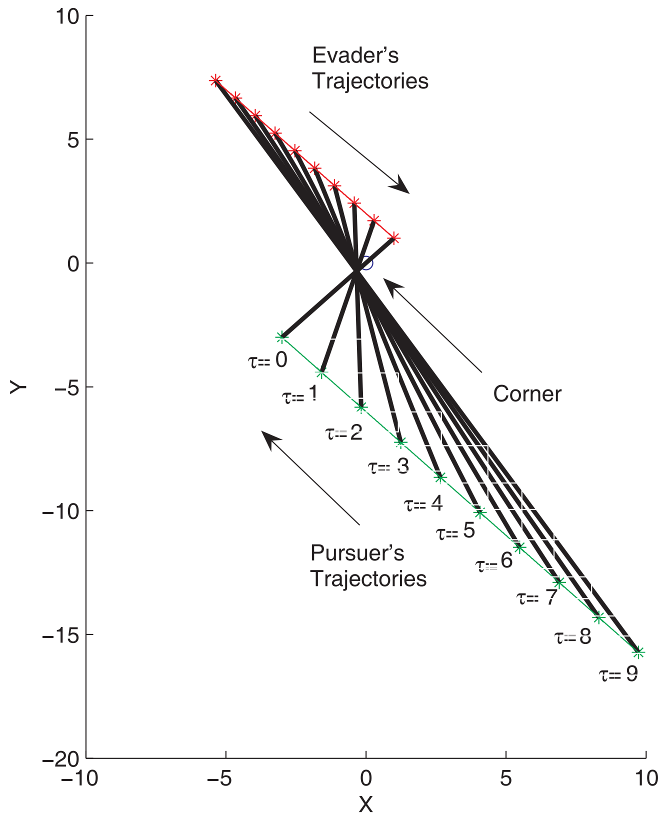

Fig. 4. Optimal trajectories for a terminating situation around a corner.

Let $(x_f, y_f, l_f, \theta_f)$ denote the configuration of the bar at the termination situation. The optimal trajectory of the pursuer as a function of retro-time is given by the following equations:

$$x_p(\tau) = x_f + \tau\overline{\upsilon}_p \sin\theta_f,$$
$$y_p(\tau) = y_f - \tau\overline{\upsilon}_p \cos\theta_f. \quad (5)$$

The optimal trajectory of the evader as a function of retro-time is given by the following equations:

$$x_e(\tau) = x_f + l_f\cos\theta_f - \overline{\upsilon}_e\tau\sin\theta_f,$$
$$y_e(\tau) = y_f + l_f\sin\theta_f + \overline{\upsilon}_e\tau\cos\theta_f. \quad (6)$$

The optimal trajectories for the pursuer and the evader are straight lines. Moreover the trajectories are perpendicular to the orientation of the bar at the termination situation and hence parallel to each other. The players move in opposite directions as they follow the optimal trajectories. Figure 4 shows the optimal trajectories for the pursuer and the evader that terminate at a corner at the origin. The evader is shown by the red dots and the pursuer is shown by green dots. The black line joining the pursuer and the evader represents the orientation of the bar (line of sight) at different time instants. The value of the speed ratio, r, is 0.5. At the termination situation, the bar is oriented at an angle of $\pi/4$ with respect to the x-axis, the position of the pursuer is $(-3, -3)$ and the position of the evader is $(1, 1)$. The payoff for both players at any point on the optimal trajectory is given by the variable τ since it is the time required for termination. In the figure, the payoff for an orientation of the bar is shown on the side of the bar. The bar with $\tau = 0$ represents the termination situation. If the pursuer deviates from its optimal strategy, then the evader has a strategy for which it can escape around the corner in time less than τ. If the evader deviates from its optimal trajectory then the pursuer has a strategy for which it can track the evader for a time greater than τ. This is due to the fact that the trajectories are obtained from strategies that are in Nash equilibrium. Hence, there is no motivation for either of the players to deviate from their optimal strategies.

For a general environment in the plane, the optimal trajectories lie in $\mathbb{R}^3 \times S^1$. In order to depict them in $\mathbb{R}^3$, we need to consider a subspace of the optimal paths terminating at a corner. In the following examples, for each corner in the environment we show the subspace of the optimal paths that have a fixed distance of the pursuer from the corner at the termination situation. The value of the speed ratio, r, is 0.66 in all of the following examples. Figure 5 shows the optimal trajectories for the players in a simple environment containing a point obstacle at the origin. The line of sight between the pursuer and the evader is broken if it passes through the origin. The evader wants to minimize the time required to break the line of sight and the pursuer wants to maximize it. Let $(x_f, y_f, l_f, \theta_f)$ represent the orientation of the bar at the termination situation. Figure 5(a) shows the optimal trajectories of the players for all possible values of l_f for a constant value of x_f, y_f and θ_f. Figure 5(b) shows the optimal trajectories for every orientation of the bar at the termination situation. The z-axis represents the angle of the bar at the termination situation. A cross-section parallel to the xy-plane gives the optimal trajectories of the players in a plane for a given θ_f. The red line in the middle denotes the point obstacle. The inner spiral is formed by the optimal trajectories of the evader and the outer spiral is formed by the optimal trajectory of the pursuer. The color of a point is a representative of the value of the game, $J(\mathbf{x})$, at that point. The value of the game increases as the color changes from blue to red.

Figure 6(a) shows a single corner in the plane. The internal angle at the corner is $2\pi/3$. Figure 6(b) shows the optimal trajectories of the players for the corner. The symmetry in the trajectories is due to the fact that termination situations occur symmetrically around a corner.

Figure 7(a) shows a regular hexagon in the plane. Figure 7(b) shows the optimal trajectories of the players for the hexagonal obstacle.

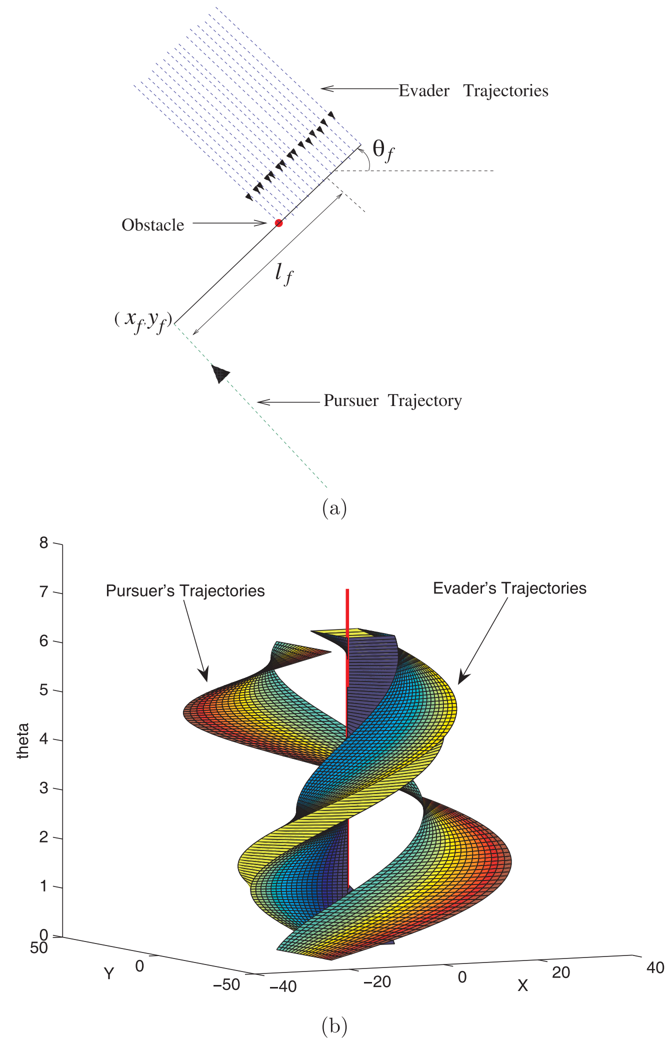

Fig. 5. Optimal trajectories for an environment having a single point obstacle: (a) optimal trajectories in the plane; (b) optimal trajectories across a section in $\mathbb{R}^3 \times S^1$.

6. Conclusion and Future work

In this paper, we have addressed a visibility-based pursuit–evasion game in an environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. The pursuer wants to maintain visibility of the evader for the maximum possible time and the evader wants to escape the pursuer's sight as soon as possible. Both players have knowledge about each others current position. Under this information structure, we present necessary and sufficient conditions for surveillance and escape. We have presented strategies for the players that are in Nash equilibrium. The strategies are a function of the value of the game. Using the strategies, we have constructed a value function by backward integration of the adjoint equations from the termination situations provided by the

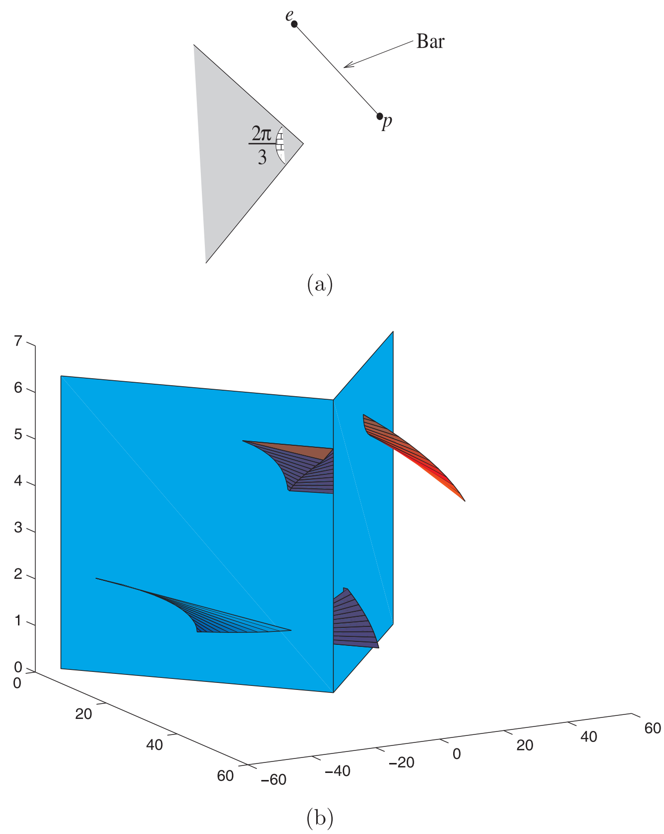

Fig. 6. Optimal trajectories of the players for a corner in space: (a) a single corner in space; (b) optimal Trajectories for the players.

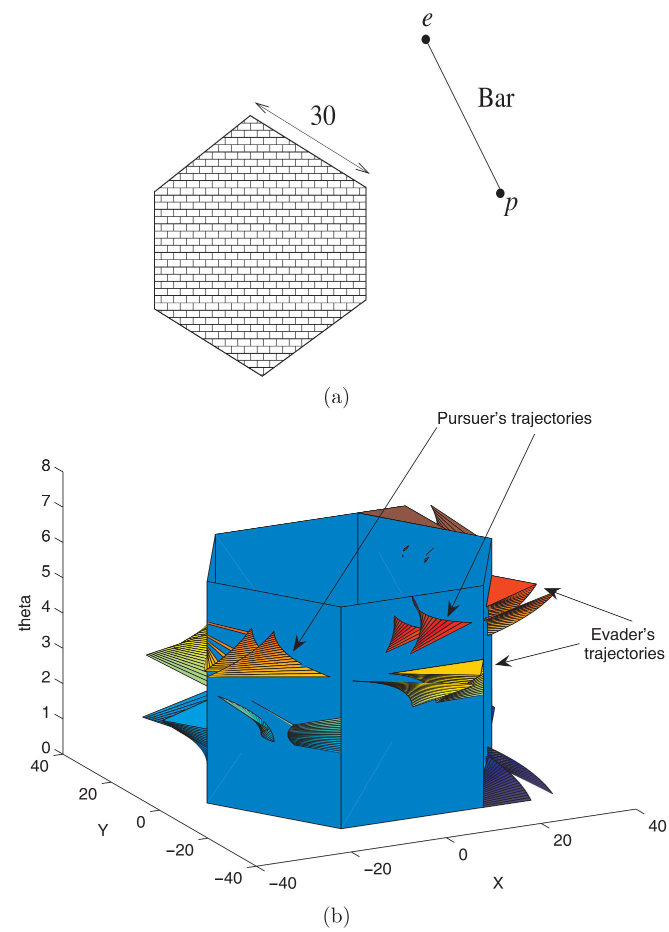

Fig. 7. Optimal trajectories of the players for a hexagonal obstacle in space: (a) a hexagonal obstacle in space; (b) optimal trajectories for the players.

corners in the environment. From the value functions we have recomputed the control strategies for the players to obtain optimal trajectories for the players near the termination situation. We have shown that the optimal strategy for the players is to move on straight lines parallel to each other in opposite directions towards a termination situation. We show a subspace of the optimal trajectories for a point obstacle, a corner and a hexagonal obstacle in space.

In order to extend the results in this paper to environment containing multiple obstacles, we plan to address the following issues in the future.

1. In a general environment there might be points from which the pursuer can see the entire free space. Such environments are called *star-shaped* and the set of points for which the property holds is called the *kernel* of the star-shaped environment. If the pursuer can reach the kernel while keeping the evader in its sight then the pursuer can see the evader forever. Hence, the kernel also provides a termination situation where the pursuer can track the evader forever. The shape of the kernel depends on the shape of the environment. At this moment, we do not have a general characterization of the shape of the kernel in the configuration space, i.e. $\mathbb{R}^3 \times S^1$ which makes it difficult to compute the final conditions for the adjoint variables in the adjoint equation. Hence, we are unable to present the optimal trajectories that are generated back from the termination situation posed due to the presence of such regions. Another interesting future direction is to efficiently compute and update the visibility polygon of a point moving in an environment (Cheng 2005). From the visibility information, we might be able to devise strategies for the pursuer to keep track of a moving evader.

2. In the previous section, we presented the optimal trajectories for the players terminating at a corner in the environment. In a general environment containing multiple obstacles, we can use the analysis of a single corner to compute globally optimal trajectories. In order to do so, we need to use suitable matching principles to gen-

erate a value function that does not violate fundamental principles of optimality. The method of singular characteristics provides us with tools to match the value functions originating from different termination situations. Our current efforts in this direction are to characterize the singular surfaces that might appear in this scenario. The construction and characterization of such surfaces analytically in four dimensions is a challenging problem. In order to overcome this difficulty we plan to use efficient numerical techniques to construct such surfaces so that we can characterize the interesting behavior of the players in the neighborhood of such surfaces.

In the future, we plan to use *viscosity* solutions to propose an algorithm to construct the optimal trajectories for the players in a environment containing multiple obstacles. We also plan to extend the results to multiple pursuers chasing an evader. In addition, we plan to extend our work to players having non-holonomic constraints in their motion.

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