

## 1 Introduction

The field of combinatorial discrepancy theory deals with the problem of coloring a set of  $T$  elements red or blue such that every set in a collection of given sets is nearly-balanced (reference books [2, 3, 1]). Formally, given a finite universe  $U := \{1, 2, \dots, T\}$  of  $T$  elements and a collection of sets  $\mathcal{S} \subseteq 2^U$  with  $|\mathcal{S}| = n$ , the combinatorial *discrepancy minimization* problem is to find a coloring  $\varepsilon \in \{+1, -1\}^U$  of the  $T$  elements to minimize

$$\max_{S \in \mathcal{S}} \left| \sum_{v \in S} \varepsilon_v \right|.$$

We will also be interested in understanding the minimum possible discrepancy

$$\text{disc}(U, \mathcal{S}) := \min_{\varepsilon \in \{+1, -1\}^U} \max_{S \in \mathcal{S}} \left| \sum_{v \in S} \varepsilon_v \right|.$$

Discrepancy theory has applications in many fields such as LP rounding, sparsification, differential privacy, and fair division. However, in today's lecture we will only focus on the techniques that are used to bound discrepancy.

### 1.1 Warm-up

To build some intuition, let's consider some simple solutions. Clearly, we have  $\text{disc}(U, \mathcal{S}) \leq T$  because we can color all elements  $+1$  and every set in  $\mathcal{S}$  contains at most  $T$  elements.

Another simple solution is to randomly color elements  $\{+1, -1\}$  independently and uniformly. Note that for any  $S \in \mathcal{S}$ , we have  $\mathbb{E}[\sum_{t \in S} \varepsilon_t] = 0$ . Moreover, by applying Chernoff bounds, we can get that  $\mathbb{P}[|\sum_{t \in S} \varepsilon_t| > c\sqrt{T \log n}] \leq \exp(-\text{poly}(n))$  for some constant  $c$ . Thus we can take a union bound over all the  $n = |\mathcal{S}|$  sets to get that with high probability  $\max_{S \in \mathcal{S}} \left| \sum_{v \in S} \varepsilon_v \right| = O(\sqrt{T \log n})$ .

Can we do better, especially when  $T \gg n$ ? For  $n = 1$  (i.e., when there is a single set in  $\mathcal{S}$ ) this is easy, because we can alternately color the elements of the set to get at most 1 discrepancy. For  $n = 2$ , note that there are only four "types" of elements: those that belong to exactly one of the sets, those that belong to both the sets, and those that belong to neither of the two sets. Consider an algorithm that alternately colors elements of the same type  $\{+1, -1\}$ . Observe that if there are even number of elements of a type, their net contribution to every set  $S \in \mathcal{S}$  is zero. Since in general there could odd number of elements of every type, the max discrepancy is at most 3 (elements that are not present in either set don't contribute). In general this approach gives a  $2^n - 1$  bound on discrepancy.

Can we obtain a  $\text{poly}(n)$  bound on discrepancy? Can we obtain an  $O(1)$  bound?

## 1.2 Linear Algebraic Formulation

To obtain sub-exponential bounds, it will be useful to formulate the discrepancy minimization problem linear algebraically. Consider the incidence matrix  $A \in \{0, 1\}^{n \times T}$  where  $A_{i,t} := 1$  [element  $t$  belongs to  $i$ -th set in  $\mathcal{S}$ ]. The discrepancy objective for color  $\varepsilon$  can now be succinctly written as  $\|A\varepsilon\|_\infty$ .

The above formulation leads us to the more general *vector balancing* problem. In this problem we are given  $T$  vectors  $a_1, \dots, a_T \in [-1, 1]^n$  and the goal is to find a coloring  $\varepsilon \in \{-1, +1\}^T$  to minimize  $\|A\varepsilon\|_\infty$ . The techniques that we will see in today's lecture will easily extend to this more general problem.

Let's start by showing that  $O(1)$  discrepancy bound is impossible for the vector balancing problem. In fact, there is an  $\Omega(\sqrt{n})$  lower bound.

**Lemma 1.** *There exist  $n$  vectors  $a_1, \dots, a_n \in [-1, 1]^n$  such that*

$$\min_{\varepsilon \in \{-1, +1\}^n} \|A\varepsilon\|_\infty \geq \sqrt{n}.$$

*Proof.* Consider the Hadamard matrix  $H_n \in \{-1, +1\}^{n \times n}$  (see Wikipedia). This matrix has the property that its  $n$  columns have length  $\sqrt{n}$  and are pairwise orthogonal. Consider the best possible coloring  $\varepsilon^*$ . Since all the columns of  $A$  are orthogonal, we have  $\|A\varepsilon^*\|_2^2 = \sum_t \|a_t\|_2^2 = n^2$ . Now using Cauchy-Schwarz, we get

$$\|A\varepsilon^*\|_\infty \geq \frac{1}{\sqrt{n}} \|A\varepsilon^*\|_2 = \sqrt{n}. \quad \square$$

We leave as an exercise to extend this proof to show an  $\Omega(\sqrt{n})$  lower bound for the discrepancy minimization problem, i.e., when each  $a_t \in \{0, 1\}^n$ . The idea is to consider the matrix  $\frac{H_n + J_n}{2}$  where  $J_n$  is the  $n \times n$  all ones matrix. See [1, Chapter 13] for details.

## 2 Reducing $T$ to $n$

In this section we will see an LP based technique that will help us reduce the problem size from  $T$  elements/vectors to  $n$  elements/vectors, and hence our trivial random coloring algorithm will already imply an  $O(\sqrt{n \log n})$  bound on discrepancy by Chernoff bounds.

### 2.1 Detour: Basic Solution

For any given a linear program  $\max c^T x$  such that  $Ax \leq b$  and  $0 \leq x_i \leq 1$ , there could be infinitely many optimal solutions. Intuitively, a basic optimal solution is an optimal solution that is at the “vertex” of the feasible region. Such a solution has the advantage that we know that several inequalities will be tight.

Formally, consider a linear program in  $T$  fractional variables  $x_1, \dots, x_T$ , with  $2T$  box constraints  $-1 \leq x_t \leq 1$  and some other  $n$  inequalities  $a_i^T x \leq b_i$ . If the given LP is feasible, a basic optimal solution is guaranteed to have at least  $T$  tight constraints. This is because  $T$  linearly independent constraints uniquely identify a  $T$  dimensional solution.

## 2.2 Obtaining an $O(\sqrt{n \log n})$ Bound

Consider the following feasibility LP for the vector balancing problem, where  $x_t \in [-1, 1]$  denotes the fractional color for vector  $a_t \in [-1, 1]^n$ .

$$-1 \leq x_t \quad \forall t \in \{1, \dots, T\} \quad (1)$$

$$x_t \leq 1 \quad \forall t \in \{1, \dots, T\} \quad (2)$$

$$\sum_t a_t(i) \cdot x_t = 0 \quad \forall i \in \{1, \dots, n\} \quad (3)$$

Consider a basic feasible solution  $x^* \in [-1, 1]^T$  for the above linear program. Our algorithm colors  $a_t$  independently  $\{+1, -1\}$  such that its expected color is  $x_t^*$ . In other words, set  $\varepsilon_t$  to  $+1$  with probability  $(1 + x_t^*)/2$ , and  $-1$  otherwise. We will show that w.h.p.  $(1 - 1/\text{poly}(n))$  this algorithm has  $O(\sqrt{n \log n})$  discrepancy.

Since  $x^*$  is a basic solution, we know that it will have at least  $T$  tight constraints. There are only  $n$  constraints in (3), so there are at least  $T - n$  variables in  $x^*$  that will be either  $-1$  or  $+1$ . All these integral variables will be colored deterministically by our algorithm, so the randomness only comes from at most at most  $n$  fractional variables. Again by standard Chernoff bounds, the probability that we incur more than  $c\sqrt{n \log n}$  discrepancy in any dimension is  $1/\text{poly}(n)$ , so we are done by union bound.

## 3 Obtaining an $O(\sqrt{n})$ Bound

We saw an  $\Omega(\sqrt{n})$  lower bound in §1.2 and an  $O(\sqrt{n \log n})$  upper bound in §2.2. In this section we will show that it's possible to remove the  $\sqrt{\log n}$  factor in the upper bound.

**Theorem 2** (Spencer). *There exists a coloring  $\varepsilon \in \{-1, +1\}^T$  for the vector balancing problem that gives  $O(\sqrt{n})$  discrepancy.*

Since we can use the idea of working with a basic solution from previous section, we will prove this theorem assuming  $T \leq n$ .

### 3.1 Partial Coloring Method

The main idea in the proof of Theorem 2 is to first find a “partial coloring” and then recurse to find a full coloring. Formally, we will show the following partial coloring lemma whose proof is deferred to §3.2. (Think of  $T \leq n$  in this lemma.)

**Lemma 3** (Partial Coloring). *There exists a coloring  $\varepsilon \in \{-1, 0, +1\}^T$  with at least  $T/10$  non-zero colors such that  $\|A\varepsilon\|_\infty = O(\sqrt{T \log(n/T)})$ .*

Next we finish the proof of  $O(\sqrt{n})$  discrepancy bound using this lemma.

*Proof of Theorem 2.* We apply Lemma 3 in rounds on the set of uncolored vectors and add the discrepancies incurred in each round. Let  $T_k$  denote the number of uncolored vectors in the  $k$ -th round. Since in each round we color at least  $1/10$  fraction of the vectors,

$T_k \leq (0.9)^{(k-1)}T \leq (0.9)^{(k-1)}n$  since  $T \leq n$ . Thus, by Lemma 3 the discrepancy in  $k$ -th round is  $c(\sqrt{T_k \log(n/T_k)})$  for some constant  $c$ . So the overall discrepancy is bounded by

$$\sum_k c\sqrt{T_k \log(n/T_k)} \leq \sum_k c\sqrt{n(0.9)^{k-1} \log\left(\frac{n}{n \cdot 0.9^{k-1}}\right)} = O(\sqrt{n}). \quad \square$$

### 3.2 Proof Idea for Partial Coloring Lemma

The original proof of the partial coloring Lemma 3 due to Spencer was based on a clever entropy based argument. Today we will discuss a method based on the Pigeonhole principle. Both these proofs are non-algorithmic, i.e., do not give a polynomial time algorithm to find the partial coloring. In recent years several polynomial time algorithms have been developed, e.g., see [1, Chapter 13] or [5].

The rest of this section is based on Nikhil Bansal's notes [4].

For any convex body  $K \in \mathbb{R}^T$ , we define its gaussian measure  $\gamma(K) := \mathbb{P}_{g \sim N(0, I_T)}[g \in K]$ . We will be repeatedly using the following basic lemma from convex geometry, whose proof we'll skip.

**Lemma 4** (Sidak-Khatri). *For any symmetric convex body  $K$  (i.e., for every  $x \in K$ , we have  $-x \in K$ ) and slab  $S := \{x : |\sum_t c_t x_t| \leq \Delta\}$ , we have  $\gamma(K \cap S) \geq \gamma(K)\gamma(S)$ .*

The main result of this section is the following powerful theorem.

**Theorem 5** (Gluskin). *Any symmetric convex body  $K$  with Gaussian measure  $\gamma(K) \geq 2^{-T/5}$  satisfies that there is a point  $\varepsilon \in \{-1, 0, +1\}^T \cap (2K)$  with at least  $n/10$  non-zero colors.*

We first finish the proof Lemma 3 using this theorem.

*Proof of Lemma 3.* Consider convex body  $K$  formed by intersection of  $n$  slabs  $|\sum_t a_t(i)x_t| \leq \Delta$ , where  $\Delta = c\sqrt{2}\sqrt{T \log(n/T)}$  for some constant  $c$ . We will use Lemma 4 to show that  $\gamma(K) \geq 2^{-T/5}$ , which will prove this lemma by Theorem 5 because any  $\varepsilon \in \{-1, 0, +1\}^T \cap 2K$  satisfies  $\|A\varepsilon\|_\infty = O(\sqrt{T \log(n/T)})$ .

Consider the slab  $S = \{x : |\sum_t a_t(i)x_t| \leq \Delta\}$ . Since Gaussian measure is the same along every direction, we can write  $\gamma(S) \approx 1 - \exp\left(-\frac{\Delta^2}{2\sum_t a_t(i)^2}\right) \leq 1 - \exp\left(-\frac{\Delta^2}{2T}\right) = 1 - \left(\frac{T}{n}\right)^c \approx \exp\left(-\frac{T^c}{n^c}\right)$ . Applying this bound for all  $i$  and using Lemma 4, we get

$$\gamma(K) \geq \prod_{i=1}^n \exp\left(-\frac{T^c}{n^c}\right) = \exp\left(-T\frac{T^{c-1}}{n^{c-1}}\right) \geq 2^{-T/5}. \quad \square$$

Finally, we given an idea for the proof of Theorem 5 based on a clever Pigeonhole argument. The proof might seem complicated because it uses some ideas from convex geometry that we did not cover in this course, but these ideas are basic.

*Proof idea for Theorem 5.* Let  $V := \{-1/2, 1/2\}^T$ , i.e., all the vertices of the unit-cube. For any  $v \in V$ , we define  $K_v = v + K := \{v + x \text{ for } x \in K\}$ . Since the length of  $v$  is  $\sqrt{T}/4$ , by a standard fact (see [4] for a proof) one can show  $\gamma(K_v) \geq \exp(-\|v\|^2) \cdot \gamma(K) = \exp(-T/4) \cdot \gamma(K)$ . Now notice that

$$\sum_{v \in V} \gamma(K_v) \geq 2^T \exp(-T/4) \gamma(K) \geq 2^{T/2}.$$

Since Gaussian measure of the entire space  $\gamma(\mathbb{R}^T) = 1$ , there exists some point  $y \in \mathbb{R}^T$  and a set  $S \subseteq V$  with  $|S| \geq 2^{T/2}$  such that  $y \in K_v$  for all  $v \in S$ .

Next we use a combinatorial result that for any set  $S \subseteq V$  with  $|S| \geq 2^{T/2}$ , there exist two “distant” elements  $z, z' \in S$  such that  $\|z - z'\|_1 \geq T/10$  (exercise: prove that  $\|z - z'\|_1 \geq \alpha T$  for some constant  $\alpha > 0$ ). We prove the theorem by setting  $\varepsilon = z - z'$ .

Clearly,  $z - z' \in \{-1, 0, 1\}^T$ . Since  $\|z - z'\|_1 \geq T/10$ , we also get that there are at least  $T/10$  non-zero colors in  $z - z'$ . Finally, we have  $z - z' \in 2K$  because we know  $y = z + k = z' + k'$  for some  $k, k' \in K$ , which implies  $z - z' = k' - k \in 2K$  since  $K$  is a symmetric body.  $\square$

## 4 Exploiting Column Sparsity

As we saw in §1.2, it’s impossible to get  $o(\sqrt{n})$  discrepancy bounds for general set families. However, it turns out that if the vectors  $a_t$  are  $s$ -sparse, i.e. the number of non-zero entries in every vector  $a_t$  is at most  $s$  (combinatorially, this translates to each element is present in at most  $s$  sets), then we can do much better. In particular, in Problem Set 4 we will prove an  $O(s)$  bound for this problem. Using ideas similar to §2.2, it’s also possible to obtain an  $O(\sqrt{s \log n})$  bound, which is incomparable since it could be better or worse depending on how large is  $n$  (compared to  $s$ ).

On the lower bound side, the proof in §1.2 already gives an  $\Omega(\sqrt{s})$  lower bound. A major open problem in Discrepancy Theory is Komlos conjecture which says that there should always be a coloring that gets  $O(\sqrt{s})$  discrepancy. Currently we don’t know anything better than  $O(s)$  for large  $n$ .

## References

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