

## Lecture The Ellipsoid Algorithm

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## 1 The Algorithm for Linear Programs

**Problem 1.** Given a polyhedron  $P$ , written as  $Ax \leq b$ , find a point in  $P$ .

Before tackling this problem, we begin with some definitions. A real symmetric matrix  $A$  with the property that  $x^T Ax > 0$  for all  $x \neq 0$  is called *positive definite*. If  $A$  is positive definite, then there exists an invertible matrix  $P$ , such that  $A = P^T P$ . Let  $D$  be a positive definite matrix and consider the ellipsoid  $\text{Ell}(D, z) = \{x : (x - z)^T D^{-1}(x - z) \leq 1\}$ . Let  $\nu$  be the maximum number of bits required to describe a vertex of  $P$  and set  $R = 2^\nu$ . To solve Problem 1 we apply the following algorithm:

### THE ELLIPSOID ALGORITHM

Start with the ellipsoid  $E_0 = \text{Ell}(R^2 I, 0)$ .

At the  $i^{\text{th}}$  iteration, check whether  $z_i$  is in  $P$ .

- YES. Output  $z_i$  as the feasible point.

- NO. Find a constraint for  $P$ ,  $a_k \cdot x \leq b_k$ , violated by  $z_i$ . Recurse on  $E_{i+1}$ , the minimum volume ellipsoid containing  $E_i \cap \{x \mid a_k \cdot x \leq a_k \cdot z_i\}$ .

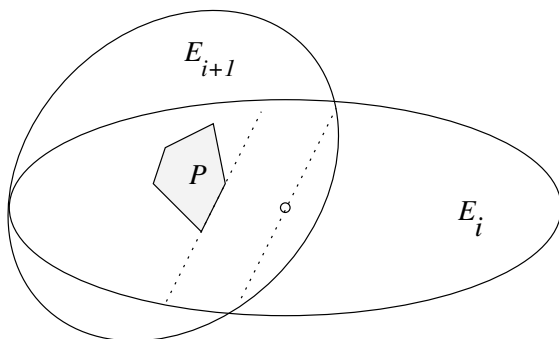


Figure 1: One cycle of the algorithm

Observe that the algorithm halts when a point  $z_i$  is found to be within  $P$ . It must halt,

since at any step  $i$ ,  $P$  is a subset of  $E_i$ , and we will see that after each step, the volume of  $E_{i+1}$  has decreased by an appreciable amount. For some value of  $i$ , the volume of  $E_i$  will be smaller than the volume of  $P$ , so the algorithm must halt before reaching this point. In the next section we show that the algorithm can actually be implemented in polynomial time.

## 2 The Time Bound

**Lemma 1.** *The minimum volume ellipsoid containing  $\text{Ell}(D, z) \cap \{x \mid a \cdot x \leq a \cdot z\}$  is exactly  $E' = \text{Ell}(D', z')$ , where*

$$z' = z - \frac{1}{n+1} \frac{Da}{\sqrt{a^T D a}} \quad (1)$$

and

$$D' = \frac{n^2}{n^2 - 1} \left( D - \frac{2}{n+1} \frac{Daa^T D}{a^T D a} \right) \quad (2)$$

and

$$\frac{\text{vol}(E')}{\text{vol}(E)} \leq e^{\frac{-1}{2n+2}} \quad (3)$$

**Sketch of proof :** First, note that  $\text{Ell}(A, 0)$  can be obtained from  $\text{Ell}(I, 0)$  (the unit ball) using the transformation  $y = Bx$ , where  $A = B^T B$ . To see this, consider the following:

$$\begin{aligned} x^T x &\leq 1 \\ y^T (B^{-1})^T (B^{-1}) y &= x^T x \\ y^T (B^{-1})^T (B^{-1}) y &\leq 1 \\ y^T A^{-1} y &\leq 1 \end{aligned}$$

where the first and last equations define the unit ball and  $\text{Ell}(A, 0)$ , respectively.

Now, first we will prove the results (1) and (2) for the special case of the unit ball,  $E = \text{Ell}(I, 0)$ . In this case, (1) reduces to

$$z' = z - \frac{1}{n+1} \frac{a}{\sqrt{a^T a}}$$

and (2) reduces to

$$D' = \frac{n^2}{n^2 - 1} \left( I - \frac{2}{n+1} \frac{aa^T}{a^T a} \right)$$

Since  $E$  is a ball, we can rotate  $a$  without affecting anything. So, assume  $a = [1, 0, \dots, 0]^T$ . Then,  $z' = [-1/(n+1), 0, \dots, 0]^T$ .

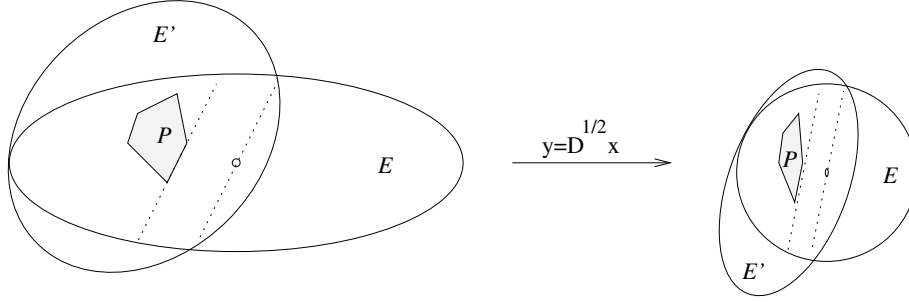


Figure 2: Transform space to take  $E$  to a ball

$$\begin{aligned}
 D' &= \frac{n^2}{n^2-1} \left( I - \frac{2}{n+1} \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \right) \\
 &= \frac{n^2}{n^2-1} \begin{bmatrix} 1 - \frac{2}{n+1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}
 \end{aligned}$$

The simplified statements for  $z'$  and  $D'$  can be proved by calculus. The general case is then proved by applying a transformation of  $A$  to the unit ball ( $B^{-1}$  above; the transformation scales the volume of a convex set by the factor  $\det(B)$ ).

Assuming (1) and (2), we can now prove (3). observe that:

$$\frac{\text{vol}(\text{Ell}(D', z'))}{\text{vol}(\text{Ell}(D, z))} = \frac{\text{vol}(\text{Ell}(I, 0))}{\text{vol}(\text{Ell}(I, 0))} \frac{\sqrt{\det(D')}}{\sqrt{\det(D)}}$$

We transform space to take  $E$  to the unit ball. The transformed  $E'$  is still the minimum ellipsoid containing half of  $E$ .

Then, assuming  $D = I$ , we have  $\sqrt{\det(D')} = \text{vol}(E')/\text{vol}(E)$ . Now we can use (2).

$$D' = \frac{n^2}{n^2 - 1} \left( I - \frac{2}{n+1} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \right)$$

The determinant of this matrix is

$$\det(D') = \left( \frac{n^2}{n^2 - 1} \right)^n \left( 1 - \frac{2}{n+1} \right)$$

Hence,

$$\begin{aligned} \frac{\text{vol}(E')}{\text{vol}(E)} &= \left( \frac{n^2}{n^2 - 1} \right)^{n/2} \left( \frac{n-1}{n+1} \right)^{1/2} \\ &= \left( \frac{n^2}{n^2 - 1} \right)^{\frac{n-1}{2}} \frac{n}{(n-1)^{1/2}(n+1)^{1/2}} \frac{(n-1)^{1/2}}{(n+1)^{1/2}} \\ &= \left( 1 + \frac{1}{n^2 - 1} \right)^{\frac{n-1}{2}} \left( 1 - \frac{1}{n+1} \right) \\ &\leq e^{\frac{1}{(n-1)(n+1)} \frac{(n-1)}{2}} e^{\frac{-1}{n+1}} = e^{\frac{-1}{2(n+1)}} \end{aligned}$$

(using  $e^x \geq 1 + x$ ). □

We need to calculate how small  $P$  can be in order to obtain a bound on the number of times we shrink  $E'$ . We will see that  $\text{vol}(P) \geq 2^{-2n\nu}$  by finding a simplex inside  $P$ . Clearly the volume of the simplex will be less than or equal to the volume of  $P$ .

Now there exist  $n + 1$  affinely independent vertices of  $P$ , say  $x_0, x_1, \dots, x_n$ .

$$\text{vol}(\text{conv}(x_0, \dots, x_n)) = \frac{1}{n!} \left| \det \begin{bmatrix} 1 & 1 & & 1 \\ & & \dots & \\ x_0 & x_1 & & x_n \end{bmatrix} \right|$$

A vertex  $x_i$  is a solution to a subset  $C_i$  of rows of  $Ax \leq b$ . We can solve for it using Cramer's Rule,  $x_{ij} = \frac{\det(C_{ij})}{\det(C_j)}$ , where  $C_{ij}$  is the matrix  $C_i$  with the  $i^{\text{th}}$  column replaced by  $b$  restricted to the relevant rows for  $C_i$ . So,

$$\text{vol}(\text{conv}(x_0, \dots, x_n)) = \frac{1}{n!} \left| \det \begin{bmatrix} 1 & 1 & & \\ \frac{\det C_{11}}{\det C_1} & \frac{\det C_{12}}{\det C_2} & \dots & \\ \frac{\det C_{21}}{\det C_1} & \frac{\det C_{22}}{\det C_2} & & \\ \vdots & & \ddots & \end{bmatrix} \right|$$

Pulling out the denominators, we see that

$$\frac{1}{n!} \left| \det \left( \begin{bmatrix} \det C_1 & \det C_2 & & \\ \det C_{11} & \det C_{12} & & \\ & & \ddots & \\ & & & \det C_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{\det C_1} & & & \\ & \frac{1}{\det C_2} & & \\ & & \ddots & \\ & & & \frac{1}{\det C_n} \end{bmatrix} \right) \right|$$

$$\geq \frac{1}{n!} \frac{1}{\det(C_1) \det(C_2) \cdots \det(C_n)}$$

As  $\det C_i \leq 2^\nu$ , we have  $\text{vol}(\text{conv}(x_0, \dots, x_n)) \geq n^{-n} (2^{-\nu})^n \geq 2^{-2n\nu}$ . After  $i$  steps,  $\text{vol}(E_i) \leq (2R)^n e^{\frac{-i}{2n+2}}$ . We stop before  $\text{vol}(E_i) < \text{vol}(P)$ . Thus,

$$2^{(\nu+1)n} e^{\frac{-i}{2n+2}} < 2^{-2n\nu}$$

which means we stop when  $i = O(n^2\nu)$ . Recall that  $\nu$  was less than the number of bits required to write down any  $n \times n$  subset of  $\{A, b\}$ , plus  $\log n$  bits. So, the the number of iterations is  $O(n^2 \langle C, d \rangle)$ . If we use  $L$ -bit numbers, then  $\langle C, d \rangle = O(n^2 L)$ . To check the validity of a point, we must check each constraint of  $P$ , taking  $O(mn)$  time. This dominates the time required to calculate the minimum ellipsoid. So the total time required to complete the algorithm is at most  $O(mn^5 L)$ .