Semi-Definite Relaxations for Minimum Bandwidth and other Vertex-Ordering problems*

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Abstract

We present simple semi-definite programming relaxations for the NP-hard minimum bandwidth and minimum length linear ordering problems. We then show how these relaxations can be rounded in a natural way (via random projection) to obtain approximation guarantees for both of these vertex-ordering problems.

1 Introduction

Let the vertices of an undirected graph be ordered $1, 2, \ldots, n$. We define the *dilation* of an edge (i, j) as the difference $|i - j|$, i.e., the length of the edge when the vertices of the graph are placed on the line in the order $1, 2, \ldots, n$.

Given an undirected graph $G = (V, E)$, we consider the following two problems:

1. *Minimum Bandwidth* : find an ordering that minimizes the maximum dilation among all the edges, i.e., minimizes
$$\max_{e \in E} \quad dilation(e).$$
2. *Minimum-length Linear Ordering* : find an ordering that minimizes the *length* (the L_2 norm) of the ordering, which is defined as:
$$\sqrt{\sum_{e \in E} dilation(e)^2}.$$

   That is, the squared length is the sum of the squares of dilations of the edges.

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We present new approximation algorithms for both these problems. Our main algorithmic tool is semi-definite programming. Using a simple semi-definite relaxation we derive an $O(\sqrt{\frac{n}{b}}\log n)$ approximation for the minimum bandwidth on an n-node graph with bandwidth b. A refinement of this relaxation allows us to get an $O(\log^2 n)$ approximation for the minimum-length linear ordering problem[1].

Recently (and independently) Feige [6] introduced the notion of a volume-respecting embedding of an undirected graph, and used it to achieve a poly-logarithmic approximation for the bandwidth problem. Interestingly there are many similarities between the two approaches. Specifically, the rounding procedure of our algorithm, projection to a random line, is also a key step in his algorithm. Our relaxation for the minimum-length ordering problem was developed after Feige's results were announced, and was inspired by his work.

Early interest in these kinds of ordering problems in the 1950's was fueled by research in the area of solvers for sparse symmetric linear systems of equations, using Gaussian elimination (such as in the finite element analysis of steel frameworks). As a heuristic to minimize the space, time and total work in the elimination procedure, it is desirable to reorder the rows (and columns) of the matrix so as to collect all the non-zero entries within a band of small width centered at the diagonal. When the (symmetric) non-zero elements of the matrix are viewed as vertex adjacencies in an undirected graph, then the reordering problem is the minimum bandwidth problem on this graph. Another application of bandwidth minimization is in search algorithms, where the minimum-bandwidth ordering of the vertices of the problem's constraint graph can be used as a branching order that reduces backtracking. For a survey on the bandwidth problem and early approaches, see [2].

The minimum bandwidth problem was first shown to be NP-hard in 1976 [15], and later even for trees of degree at most three and for caterpillars [7, 13]. Approximations algorithms have been known only for some special families of graphs, such as caterpillars or asteroidal triple-free graphs [11, 12].

2 The Semi-Definite Relaxation

Our approximation algorithm begins with an SDP (Semi-Definite Programming) relaxation. First we motivate and describe the relaxation for the bandwidth problem and then the relaxation for the minimum-length linear ordering problem.

Semidefinite programming is the optimization of a linear function over variables drawn from a symmetric matrix subject to linear constraints on these variables and the added requirement that this symmetric matrix be positive semidefinite. Semidefinite programs can be solved to within an arbitrarily small additive error in polynomial-time [8] - for more background, see the description in [9]. Our semidefinite relaxation for the bandwidth problem is inspired by the pioneering work of Goemans and Williamson [9] on the Maximum Cut problem, and is based on the following observation that their work motivates and uses: An $n \times n$ symmetric positive semidefinite matrix A can be decomposed into $A = B^T B$ where B is an $m \times n$ matrix of full row rank (hence $m \leq n$) via an incomplete Cholesky factorization (see, e.g., [10]). Moreover, if we insist that $a_{ii} = d$ for all i, then the matrix A corresponds via the decomposition to a set of equal-length vectors $b_1, b_2, \ldots, b_n$ in $\boldsymbol{R}^m$

[1]The preliminary version of this paper [1] erroneously claimed an $O(\log^{\frac{3}{2}} n)$-approximation ratio for this problem. We thank Seffi Naor [16] for bringing the error in the analysis to our attention.

(each of Euclidean length $\sqrt{d}$), namely, b_i is the i^{th} column of B, where $a_{ij} = b_i \cdot b_j$ (the dot product of b_i and b_j). This allows us to view the solution to a SDP equivalently as a set of vectors in $\boldsymbol{R}^m$ for some $m \leq n$, obeying some extra linear constraints that we stipulate on the dot products of these vectors.

2.1 Bandwidth

The minimum bandwidth problem on an undirected graph $G = (V, E)$ is equivalent to the following geometrical embedding problem: Assign the nodes of the graph to distinct equi-spaced points along a quarter-circle of radius n in the positive quadrant of a 2-dimensional plane, such that the maximum value of the Euclidean length of any edge of the graph is minimized. The projection of this ordering on either of the co-ordinate axes bounding this quadrant recovers the optimal bandwidth ordering since the objective functions (Euclidean distance in the quarter-circle and linear distance in the line) are monotonically related. The following is then a non-convex quadratic programming formulation for the bandwidth problem. We use $\hat{x}$ and $\hat{y}$ to denote the basis vectors along the x and y co-ordinates respectively. Also $|v|$ is used to denote the Euclidean length of a vector v.

$$\begin{aligned} \text{minimize} \quad & b && (1) \\ |v_i| \;&=\; n \quad \forall i \in \{1, \ldots, n\} && (2) \\ |v_i - v_j| \;&\leq\; b \quad \forall (i, j) \in E && (3) \end{aligned}$$

and the v_i's are two-dimensional vectors each distinctly assigned to a point of the form $n \cos(\frac{\pi j}{2n})\hat{x} + n \sin(\frac{\pi j}{2n})\hat{y}$ for some positive j.

The last set of constraints cannot be enforced if we wish to carry out the optimization in polynomial time, so we relax the dimensionality of the vectors v_i and add more constraints to strengthen the resulting program. The extra constraints we add are "spreading" constraints, in the spirit of Even, Naor, Rao and Schieber [5]. With only constraints such as (2) and (3), the program will simply produce one single vector as its solution to all the v_i. We want instead that the vectors be spread out as in a line. For instance, on a line, for any point p there are at most $2k$ other points within distance k of p. We add some new constraints ((8) in the formulation below) to enforce this condition. We finally arrive at the following semidefinite programming formulation,

$$\begin{aligned} \text{minimize} \quad & b && (4) \\ v_i \cdot v_j \;&\geq\; 0 \quad \forall i, j \in \{1, \ldots, n\} && (5) \\ |v_i| \;&=\; n \quad \forall i \in \{1, \ldots, n\} && (6) \\ |v_i - v_j| \;&\leq\; b \quad \forall (i, j) \in E && (7) \\ \frac{1}{|S|} \sum_{j \in S} (v_i - v_j)^2 \;&\geq\; \frac{1}{6} \left(\frac{|S|}{2} + 1 \right) (|S| + 1) \quad \forall S \subseteq \{1, \ldots, n\},\ \forall i \in \{1, \ldots, n\} && (8) \end{aligned}$$

The goal of the above constraints is to enforce a near-linear embedding of the vertices while minimizing the value of b, which is the maximum dilation of any edge in the relaxation.

Formally, constraint set (7) states that for any edge in the graph, the distance between the corresponding vectors should be at most the optimal bandwidth. From our discussion above, it is perhaps easier to see that (7) is a legal SDP constraint if we rewrite it as $(v_i - v_j) \cdot (v_i - v_j) \le b^2$.

Constraints (5) are primarily for ease of analysis. Constraints (8) are the spreading constraints motivated earlier. These constraints are useful in the analysis of the rounding algorithm where we bound the probability that two given points in the vector representation of the solution fall into an interval of fixed width in a random projection to a line (i.e., when all the points are projected to their dot product with a random unit vector[2] passing through the origin). This probability is inversely proportional to the distance between the points (Lemma 5) and the spreading constraints allow us to upper bound this probability effectively.

Although there are exponentially many constraints in (8), it is not hard to construct a separation oracle for them, and hence the SDP can be solved in polynomial time (see Grötschel, Lovász, Schrijver [8]). To answer the separation problem for (8) for a given node i, simply sort the vertices $j \neq i$ in increasing order of $(v_i - v_j)^2$ and check for violation each of the $n-1$ sets that occur as prefixes in this order. It is easy to see that if any set S violates (8) for vertex i, then the prefix of vertices in this order of size $|S|$ also violates (8) for i. This is also the same separation oracle used by Even et al. [5] in their work on spreading metrics.

Let us refer to the above formulation as the *bandwidth* SDP. Suppose b^* is the optimal bandwidth. Then by lifting the optimal bandwidth ordering to the equi-spaced embedding in the quarter-circle described above, it is easy to verify that all the constraints are satisfied to give an objective function value of at most $\frac{\pi}{2} b^*$.

2.2 Minimum-length ordering

The SDP formulation for the Minimum-length ordering problem is similar to the bandwidth SDP. Technically, the analysis in this case will require bounding the probability that three points fall into a fixed width interval in a random projection (instead of just two points as in the bandwidth analysis). This probability can be shown to be inversely proportional to the area of the triangle formed by the three points (Lemma 6). For this reason, we add constraints in this case that lower bound the areas of triangles formed by points in the vector representation of the solution. Specifically, if we let $A(i,j,k)$ denote the area of the triangle formed by v_i, v_j, v_k, our SDP relaxation of the minimum-length linear ordering problem is the following. (The fact that it is indeed a relaxation will be established in Lemma 8.)

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} (v_i - v_j)^2 \\
v_i \cdot v_j \quad & \ge \quad 0 \quad \forall i,j \in \{1,\ldots,n\} \\
\frac{1}{|S|}\sum_{j\in S}(v_i - v_j)^2 \quad & \ge \quad \frac{1}{6}\left(\frac{|S|}{2}+1\right)(|S|+1) \quad \forall S \subseteq \{1,\ldots,n\},\ \forall i \in \{1,\ldots,n\}
\end{aligned}$$

[2]Here and henceforth, we use the term "unit vector" to denote a vector of unit length, not only those along the co-ordinate axes.

$$\frac{1}{|S|}\sum_{k\in S} A^2(i,j,k) \quad \geq \quad \epsilon|v_i - v_j|^2|S|^2 \quad \forall S \subseteq \{1,\ldots,n\},\ \forall i,j \in \{1,\ldots,n\} \tag{9}$$

The first two sets of constraints are identical to (5), (8) above. Instead of constraining the length of each individual edge as in (7), we minimize the squared length of the ordering (sum of squares of edge lengths). This is a linear function of the $v_i \cdot v_j$. The constraint set (9) was motivated earlier and will be useful in the analysis; ϵ is a constant greater than 0 that can be calculated from Lemma 11 in the Appendix. We address below the incorporation of constraints (9) in the semidefinite formulation.

Fact 1 *For two vectors v_i, v_j, the square of the area of the triangle they form with the origin is given by*

$$\frac{1}{4}\begin{vmatrix} v_i \cdot v_i & v_i \cdot v_j \\ v_i \cdot v_j & v_j \cdot v_j \end{vmatrix}.$$

Hence, for any three vectors, v_i, v_j, v_k, the area $A(i,j,k)$ of the triangle formed by them, which is the same as the area of the triangle formed by $v_j - v_i, v_k - v_i$ and the origin, can be computed using

$$A^2(i,j,k) = \frac{1}{4}\begin{vmatrix} (v_j - v_i)\cdot(v_j - v_i) & (v_j - v_i)\cdot(v_k - v_i) \\ (v_j - v_i)\cdot(v_k - v_i) & (v_k - v_i)\cdot(v_k - v_i) \end{vmatrix}.$$

Further, the constraint, $A^2(i,j,k) \geq c$ for a real number c is a convex constraint. Note that for an arbitrary matrix X, the constraint $DET(X) \geq c$ may not be convex; however when X is restricted to being positive semi-definite (as in our case), it becomes convex. (See, e.g., [14], pp. 239.)

3 The Algorithm

Given an undirected graph $G = (V, E)$, the approximation algorithm for both problems is as follows. The only difference between the two problems is in the SDP formulations.

1. Solve the SDP relaxation for G. Let the solution obtained be $v_1, \ldots, v_n$.
2. Pick a random line through the origin, i.e., a random unit vector ℓ.
3. Project $v_1, \ldots, v_n$ on to the line ℓ.
4. Output the vertex-ordering along this line, i.e., in increasing values of $v_i \cdot \ell$.

We show that the algorithm with the bandwidth SDP finds an ordering of bandwidth at most $O(\sqrt{\frac{n}{b}}\log n)$ times the optimum with high probability. For the minimum-length ordering problem we will show that this algorithm gives an ordering of length at most $O(\log^2 n)$ times the optimum, with high probability.

3.1 Overview of Bandwidth analysis

The outline of the analysis for the Bandwidth performance guarantee is as follows. Imagine slicing up the ball of radius n into strips orthogonal to ℓ of width $b/\sqrt{n}$. The first claim is that with high probability, no edge in G crosses more than $O(\sqrt{\log n})$ strips. The reason is that for any edge (i,j) we have $|v_i - v_j| \leq b$ (by constraint 7) and since ℓ was chosen randomly, with high probability we have $|(v_i - v_j) \cdot \ell| \leq c|v_i - v_j|\sqrt{\log n}/\sqrt{n}$ (i.e., the vector $v_i - v_j$ is "nearly orthogonal" to the line ℓ; see Lemma 3). So, to prove an $\tilde{O}(\sqrt{n})$ approximation for the minimum bandwidth it suffices to prove that with "reasonable" probability, every strip has at most $\tilde{O}(\sqrt{n})$ points inside.

For a given strip s (say, the strip corresponding to the interval $[ib/\sqrt{n}, (i+1)b/\sqrt{n}]$ on line ℓ), the probability over the choice of ℓ that a given point $v \in G$ falls into s is at most $O(b/n)$. (This is because there are $O(n\sqrt{n}/b)$ strips total, and the middle n/b of them roughly equally divide up most of the probability.) Thus, the *expected* number of points in any given strip is only $O(b)$.

What about the variance? To calculate this we need to upper-bound the probability that a given pair of points v_i, v_j both fall into a given strip s. This is roughly equal to $\Pr[v_i \text{ falls into } s] \cdot \Pr[|v_i - v_j| \cdot \ell \leq b/\sqrt{n}]$. The first quantity, as described above, is $O(b/n)$, while the latter quantity is $O(b/d)$ if $|v_i - v_j| \leq d$. At this point, we use constraints (8) to show that there cannot be too many pairs of points v_i, v_j that are too close together. This allows us to bound the variance which then yields the final results. For slightly improved bounds, we use strips of width $\Theta(b\sqrt{\log n}/\sqrt{n})$ instead of $b/\sqrt{n}$.

3.2 Overview of Minimum-length Ordering analysis

The analysis is very similar to the analysis of the bandwidth guarantee. The first step is to show that the optimal ordering can be turned into a solution for the SDP formulation without much worsening in the objective function. We do this in Lemma 8 by using a lifted embedding of the optimal solution in $\log n$-dimensions so as to obey the triangle lower-bound constraints.

The proof of the rounding guarantee relies on bounding the expected value of the different terms in the objective function, one for each edge (i,j) in the graph G. This term for the edge (i,j) is the square of the number of points that fall between i and j in the random projection. Algebraic simplification shows that what is required to be bounded is the probability that a pair of other points k and l both fall between i and j in the random projection. As before, since ℓ was chosen randomly, with high probability we have $|(v_i - v_j) \cdot \ell| \leq c|v_i - v_j|\sqrt{\log n}/\sqrt{n}$, and so we are left to bound the probability that the three vectors v_i, v_k and v_l all fall in an interval of width $c|v_i - v_j|\sqrt{\log n}/\sqrt{n}$. We accomplish this by relating this inversely to the area of the triangle $A(i,k,l)$ formed by these vectors and using the lower bounds on the triangle areas.

We present the formal analyses in the next section.

4 Approximation Guarantees

We start with a useful lemma about any set of vectors satisfying the constraint set (8).

Lemma 1 *Let $v_1, \ldots, v_n \in \boldsymbol{R}^n$ satisfy the constraint set (8). Then for any ball B of radius $r \geq 1$ in $\boldsymbol{R}^n$ (not necessarily centered at the origin)*

$$|B \cap \{v_1, \ldots, v_n\}| \leq O(r).$$

Proof. Let $S = \{j : v_j \in B\}$ and suppose for contradiction that $|S| > 7r$. Pick an arbitrary $i \in S$. By constraint set (8), the average value of $(v_i - v_j)^2$ over $j \in S$ is at least $\frac{1}{6}(|S|/2+1)(|S|+1) > \frac{1}{6}(7r/2)(7r) > 4r^2$. But, this is clearly impossible because the maximum value of $|v_i - v_j|$ is at most the diameter $2r$. □

Next, we make a few observations regarding random projections.

Lemma 2 *Let $v_1, v_2, v_3 \in \boldsymbol{R}^n$. Let ℓ be a random unit vector. Let $y_i = v_i \cdot \ell$. Let θ be the angle between the vectors $(v_2 - v_1)$ and $(v_3 - v_1)$. Then the probability that y_1 lies between y_2 and y_3 is exactly θ/π.*

Proof. The probability that v_1 when projected to ℓ falls in between the projections of v_2 and v_3 is

$$\Pr[v_2 \cdot \ell \leq v_1 \cdot \ell \leq v_3 \cdot \ell] + \Pr[v_3 \cdot \ell \leq v_1 \cdot \ell \leq v_2 \cdot \ell]$$

which is the same as

$$\Pr[((v_1 - v_2) \cdot \ell)((v_3 - v_1) \cdot \ell) \geq 0]$$

which is exactly the angle between the vectors $(v_1 - v_2)$ and $(v_3 - v_1)$ divided by π. □

Fact 2 *The volume of the n-dimensional ball of radius r is equal to $\frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$ and its surface area is $\frac{2r^{n-1}\pi^{n/2}}{\Gamma(n/2)}$.*

Here $\Gamma(x)$ is Euler's Gamma function; for a positive integer x, $\Gamma(x) = (x-1)!$.

Lemma 3 *Let $v \in \boldsymbol{R}^n$. For a random unit vector ℓ,*

$$\Pr\left[|v \cdot \ell| \leq \frac{c}{\sqrt{n}}|v|\right] \geq 1 - e^{-c^2/4}.$$

Proof. The desired probability is the surface of a central band of thickness $2c/\sqrt{n}$ on a unit sphere, divided by the surface area of the whole sphere. Denote the surface area of the n-dimensional sphere of radius r by $A_n(r)$. Then the area of the region outside the central band is less than the area of an n-dimensional sphere of radius $\sqrt{1 - c^2/n}$ (since the remaining portions of the unit ball after slicing out the central band is a convex body that can be inscribed in a ball of the smaller radius). Using $A_n(r) = K_n r^{n-1}$, for K_n as in fact 2, we can lower bound the area of the central band as the area of the unit sphere minus the area of a sphere of radius $\sqrt{1 - c^2/n}$. Thus the desired probability is at least

$$\begin{aligned} \frac{A_n(1) - A_n\left(\sqrt{1 - \frac{c^2}{n}}\right)}{A_n(1)} &= 1 - \left(1 - \frac{c^2}{n}\right)^{n/2} \\ &\geq 1 - e^{-c^2/4}. \end{aligned}$$

□

Lemma 4 *Let $v \in \boldsymbol{R}^n$. For a random unit vector ℓ,*

$$\Pr[|v \cdot \ell| \leq \frac{1}{c\sqrt{n}}|v|] = O\left(\frac{1}{c}\right).$$

Proof. The desired probability can be upper-bounded by $4/c\sqrt{n}$ times the surface area of the $(n-1)$-dimensional unit ball, divided by the surface area of the n-dimensional unit ball. The factor of 4 is due to loosely upper-bounding the curvature of the n-dimensional ball within a width of $1/c\sqrt{n}$ in both directions above and below the origin. This is at most

$$\frac{4A_{n-1}}{c\sqrt{n}A_n} \leq \frac{4}{c\sqrt{n\pi}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \leq O\left(\frac{1}{c}\right).$$

□

We consider the following event: two points x, y on the surface of the ball of radius n, at a distance d from each other are projected on to a random line. What is the probability that x and y fall in any fixed interval of width W on the line? The following lemma, crucial to our analysis, bounds this probability.

Lemma 5 *Let x, y be arbitrary vectors of length n in $\boldsymbol{R}^n$ such that $|x - y| = d$ and $x \cdot y \geq 0$. Let ℓ be a random unit vector. Then, for any fixed α and width W,*

$$\Pr[\alpha \leq x \cdot \ell,\ y \cdot \ell \leq \alpha + W] = O\left(\frac{W^2}{d}\right).$$

Proof. For convenience, rotate the sphere so that

$$x = (-d/2, \sqrt{n^2 - d^2/4}, 0, \ldots)$$

and

$$y = (d/2, \sqrt{n^2 - d^2/4}, 0, \ldots).$$

Let vector $v = y - x = (d, 0, \ldots)$, and let $\ell = (\ell_1, \ell_2, \ldots)$ be our randomly chosen unit vector. Note that in order for the event in question to occur, it must be the case that $|v \cdot \ell| \leq W$. Therefore,

$$\Pr[\alpha \leq x \cdot \ell,\ y \cdot \ell \leq \alpha + W] \quad \leq \quad \Pr[|v \cdot \ell| \leq W] \cdot \Pr[\alpha \leq x \cdot \ell \leq \alpha + W \mid |v \cdot \ell| \leq W].$$

Since $|v \cdot \ell| = |\ell_1| \cdot d$, we have $\Pr[|v \cdot \ell| \leq W] = \Pr[|\ell_1| \leq W/d]$, which is $O(W\sqrt{n}/d)$ by Lemma 4.

Given the event that $|\ell_1| \leq W/d$, the inequality $\alpha \leq x \cdot \ell \leq \alpha + W$ can be relaxed to $\alpha - W/2 \leq x' \cdot \ell' \leq \alpha + 3W/2$, where x' and ℓ' are $n-1$-dimensional vectors consisting of the last $n-1$ components of x and ℓ. Since $x_i = 0$ for all $i > 2$, this is equivalent to

$$\alpha - W/2 \leq \ell_2\sqrt{n^2 - d^2/4} \leq \alpha + 3W/2.$$

The probability of this last event can be upper-bounded by computing the area of the largest possible strip of this width W (the one centered around the equator). By assumption, $x \cdot y \geq 0$, implying that $d \leq n\sqrt{2}$, so $\sqrt{n^2 - d^2/4} \geq n/\sqrt{2}$. We can now bound the fraction of the sphere covered by this strip by $O(W/\sqrt{n})$ as in the proof of Lemma 4. Thus, we finally get

$$\begin{aligned}
\Pr[\alpha \leq x \cdot \ell,\ y \cdot \ell \leq \alpha + W] &= O\left(\frac{W\sqrt{n}}{d} \cdot \frac{W}{\sqrt{n}}\right) \\
&= O\left(\frac{W^2}{d}\right).
\end{aligned}$$

□

The following lemma will be useful in the analysis for the minimum length ordering problem.

Lemma 6 *Let v_1, v_2, v_3 be vectors in $\boldsymbol{R}^n$. Then on projection to a random line, the probability that there exists an interval of width W that all three fall into is*

$$O\Big(\frac{W^2 n}{A(1,2,3)}\Big)$$

Proof. Consider the triangle $v_1v_2v_3$. Assume without loss of generality that its smallest angle is the one at v_3, and that $|v_1 - v_3| \leq |v_2 - v_3|$. Notice that the event in question is invariant under translation of the space; thus we may also assume without loss of generality that v_3 is the origin.

In order for all three points to fall into an interval of width W, it must be the case that v_1 and v_2 both fall into the interval $[-W, W]$. We bound the probability of the latter event using Lemma 5. Specifically, let $v_1' = nv_1/|v_1|$, let $v_2' = nv_2/|v_2|$, and let $d' = |v_2' - v_1'|$. The event that v_1 and v_2 both fall into the interval $[-W, W]$ implies the event that v_1' and v_2' both fall into the interval $[-Wn/|v_1|, Wn/|v_1|]$ since $|v_1| \leq |v_2|$. Since v_1' and v_2' are both length n (and $v_1' \cdot v_2' \geq 0$ by the assumption that the smallest angle is at v_3), Lemma 5 bounds the probability of this event by

$$O\left(\frac{W^2 n^2}{|v_1|^2 d'}\right).$$

Since v_3 is the smallest angle of the triangle $v_1v_2v_3$, the area of $v_1v_2v_3$ is at most twice the area of $v_1v_2''v_3$ where $v_2'' = v_2|v_1|/|v_2|$. This area equals $(|v_1|/n)^2$ times the area of $v_1'v_2'v_3$, and that area is at most $nd'/2$. Thus, $A(1,2,3) \leq |v_1|^2 d'/n$, and the desired probability is

$$O\left(\frac{W^2 n^2}{|v_1|^2 d'}\right) = O\left(\frac{W^2 n}{A(1,2,3)}\right).$$

□

4.1 Bandwidth

We begin with the following lemma.

Lemma 7 *Suppose $v_1, \ldots, v_n$ satisfy the constraint set (5), (6), and (8). For a random line ℓ, let X be the random variable denoting the number of points v_i whose projection onto ℓ falls into a given interval I of width W. Then,*

$$\mathbf{E}[X] = O(W\sqrt{n}) \quad \textit{and} \quad \mathbf{E}[X^2] = O(W^2 n \log n).$$

Proof. Define X_i to be the random variable that is 1 if the projection of v_i onto ℓ falls in I and 0 otherwise. Then from Lemma 4, $\mathbf{E}[X_i] = O(W/\sqrt{n})$, which implies

$$\mathbf{E}[X] = O(W\sqrt{n}).$$

Now consider pairs v_i, v_j. By Lemma 5 we have $\mathbf{E}[X_iX_j] = O(W^2/d_{ij})$, where $d_{ij} = |v_i - v_j|$. Therefore,

$$\begin{aligned}\mathbf{E}[X^2] &= \mathbf{E}[\sum X_i^2 + 2\sum X_iX_j] \\ &= O\Big(W\sqrt{n} + \sum_{i,j} \frac{W^2}{d_{ij}}\Big) \\ &= O(W^2 n \log n),\end{aligned}$$

where the last line follows from Lemma 1, since Lemma 1 implies that for any fixed i, $\sum_j 1/d_{ij} = O(\log n)$. □

Theorem 1 *The algorithm finds an ordering whose bandwidth is at most $O(\sqrt{n/b}\log n)$ times the minimum bandwidth with high probability.*

Proof. Let $v_1, \ldots, v_n$ be the set of vectors of length n found by solving the SDP.

First, using Lemma 3 we have that *every* edge of G, of length at most b in the SDP, when projected down to a random line has length no more than $W = 8b\sqrt{\log n}/\sqrt{n}$ with high probability.

Let ℓ be a random line and partition ℓ into intervals of width W. Using Lemma 3 one more time, with high probability, all vertices on projection fall within the middle n/b intervals (since these have total width $8\sqrt{n \log n}$). Since each edge spans at most two intervals (with high probability), it suffices now just to prove that with reasonable probability, none of these n/b intervals has more that $O(\sqrt{nb}\log n)$ vertices that project into it.

At this point we simply use Lemma 7. By Lemma 7, the random variable X denoting the number of vertices that on projection fall into a given interval of width W satisfies $\mathbf{E}[X^2] = O(W^2 n \log n)$. Therefore, by Chebychev's inequality

$$\begin{aligned}\frac{b}{4n} &\geq \Pr[X > \sqrt{4n/b}\sqrt{W^2 n \log n}] \\ &= \Pr[X > 16\sqrt{bn}\log n].\end{aligned}$$

Thus, with reasonable probability (3/4), each of the n/b intervals has only $O(\sqrt{nb}\log n)$ vertices that project into it, proving the theorem. □

4.2 Minimum-length ordering

Let $e = (i, j) \in E$, and upon projection to a random line, let Y_{ij} be the random variable whose value is the dilation of e in the ordering on the line, i.e., the number of points that fall in the span of the edge.

First, we use Lemmas 1 and 2 to show that the expectation of Y_{ij} is at most $|v_i - v_j| \log n$ as follows. Let P denote the center of the geodesic joining v_i and v_j, and let $l = |v_i - v_j|$. Consider balls with radii l, $2l$, $4l, \ldots$ centered at P. Let x_k be at distance rl from P. By lemma 2 the probability that x_k is projected between x_i and x_j is at most $\frac{2}{r}$. By lemma 1, the number of points at a distance of between rl and $2rl$ from P is $O(rl)$. So the expected number of vertices that fall in between the projections of x_i and x_j is at most

$$\sum_{m=1}^{\frac{1}{2}\log n} O(2^m l)\frac{1}{2^m} = O(l \log n).$$

However we need to bound the second moment, $\mathbf{E}(Y_{ij}^2)$. For this we need to bound the probability that a pair of vertices falls in the span of e. Lemma 6 bounds this probability as at most 1 over the area of the triangle formed by the two points and any one of the endpoints of the edge. So, on the whole we would like to ensure that the sum of the inverse areas of the triangles formed by every pair with one endpoint of e is small. This is precisely what the triangle constraints (9) achieve. Just the spreading constraints (8) do not suffice for this.

Below we describe this formally. First we show that the SDP is indeed a near-relaxation (there exists a solution to the SDP with value $\leq OPT\sqrt{\log n}$). Then we give the approximation guarantee for the rounding step.

Lemma 8 *Let OPT be the value of the minimum length ordering, and OPT_{SDP} be the objective value found by the SDP. Then,*

$$OPT_{SDP}^2 \leq OPT^2 \log n$$

Proof. Without loss of generality, let $0, \ldots, n-1$ be the minimum length ordering of G. Let the value of this ordering be OPT, i.e.,

$$OPT = \sqrt{\sum_{(i,j)\in E} (i-j)^2}$$

We will now construct an embedding of the vertices as vectors $u_0, \ldots, u_{n-1} \in \boldsymbol{R}^{\lfloor \log n \rfloor + 1}$ such that

$$|u_i - u_j| \leq |i-j|\sqrt{\lfloor \log n \rfloor + 1}$$

and further $u_0, \ldots, u_{n-1}$ satisfy the constraints of the minimum length ordering SDP. The lemma follows from these facts.

First, an example. For n=17 points, the embedding is:

$$\begin{gathered}
(0,0,0,0,0)\\
(1,1,1,1,1)\\
(2,2,2,2,0)\\
(3,3,3,1,1)\\
(4,4,4,0,0)\\
(5,5,3,1,1)\\
(6,6,2,2,0)
\end{gathered}$$

$$
\begin{array}{c}
(7,7,1,1,1)\\
(8,8,0,0,0)\\
(9,7,1,1,1)\\
(10,6,2,2,0)\\
(11,5,3,1,1)\\
(12,4,4,0,0)\\
(13,3,3,1,1)\\
(14,2,2,2,0)\\
(15,1,1,1,1)\\
(16,0,0,0,0)
\end{array}
$$

The first coordinate is just i. The second coordinate is i for $i \leq n/2$ and $n-i$ after that. The third coordinate goes up to $n/4$, down to zero, back up to $n/4$ and back down to zero again. And so on.

In general, let d be the smallest integer such that $2^d > n$. Then i is mapped to

$$(i, |i \bmod 2^{d-1} - 2(i \bmod 2^{d-2})|, \ldots, |i \bmod 2^{d-l+1} - 2(i \bmod 2^{d-l})|, \ldots, i \bmod 2).$$

That is, the l^{th} coordinate of u_i is $|i \bmod 2^{d-l+1} - 2(i \bmod 2^{d-l})|$, for $l = 1, \ldots, d$.

Since the lth coordinate of u_i differs from the lth coordinate of u_j by at most $|i-j|$, we have $(u_i - u_j)^2 \leq d(i-j)^2$. So, we have $|u_i - u_j| \leq |i-j|\sqrt{\lfloor \log n \rfloor + 1}$ as desired. Constraints (8) are satisfied because the construction of the first coordinate ensures that for any i, j, $|u_i - u_j| \geq |i-j|$.

Finally, we just need to show that constraints (9) are satisfied. This follows from the fact, given as Lemma 11 in the appendix, that for any $i < j < k$ the area of the triangle formed by u_i, u_j, u_k is $\Omega(|j-i||k-j|)$.

These observations imply that $u_0, \ldots, u_{n-1}$ satisfy the SDP, and their objective value is $O(OPT\sqrt{\log n})$. □

Let $v_1, \ldots, v_n$ be the set of vectors found by solving the SDP.

Lemma 9

$$\mathbf{E}(Y_{ij}^2) = O(|v_i - v_j|^2 \log^3 n)$$

Proof. Fix some edge (i,j). Define the random variable X_k for each $k = 1, \ldots, n$, $k \neq i, j$ to be 1 if on random projection v_k is projected in between v_i and v_j (falls in the span of the edge) and 0 otherwise. Then

$$Y_{ij} = \sum_{k \neq i,j} X_k$$

and

$$
\begin{aligned}
\mathbf{E}(Y_{ij}^2) &= \sum_{k \neq i,j} \mathbf{E}(X_k^2) + \sum_{k,l \neq i,j} \mathbf{E}(X_k X_l) \\
&= \sum_{k \neq i,j} \mathbf{E}(X_k) + \sum_{k,l \neq i,j} Pr[k, l \text{ fall between } i, j]
\end{aligned}
$$

$$\leq \mathbf{E}(Y_{ij}) + \sum_{k,l \neq i,j} Pr\left[k, l, i \text{ fall in an interval of width } \frac{4\sqrt{\log n}}{\sqrt{n}}|v_i - v_j|, \text{ or } i, j \text{ do not}\right]$$

$$\leq |v_i - v_j| \log n + 1 + \sum_{k,l \neq i,j} \frac{16 \log n |v_i - v_j|^2}{A(k,l,i)} \qquad \text{(by Lemmas 3 and 6)}$$

$$= O(|v_i - v_j|^2 \log^3 n).$$

The last step above follows from the constraint set (9) as follows:

$$\sum_{k,l \neq i} \frac{1}{A(k,l,i)} = \sum_{k \neq i} \sum_{l \neq i,k} \frac{1}{A(i,k,l)}$$

For each pair i, k the inner sum is $O((\log n)/|v_k - v_i|)$. To see this, order the remaining vertices according to their distance from i (say) and then the constraints imply that the triangle induced by the p^{th} point in this order has area at least $\Omega(p|v_i - v_k|)$. Hence

$$\begin{aligned} \sum_{k \neq i} \sum_{l \neq i,k} \frac{1}{A(i,k,l)} &\leq c \sum_{k \neq i} \frac{1}{|v_i - v_k|} \sum_{1 \leq p \leq n} \frac{1}{p} \\ &\leq c \log n \sum_{k \neq i} \frac{1}{|v_i - v_k|} \\ &= O(\log^2 n). \end{aligned}$$

Here c is a constant. The last step is implied by the constraint set (8). $\square$

Theorem 2 *The expected length of the ordering found by the algorithm is $O(\log^2 n)$ times the optimum.*

Proof. The expected value of the square of the length of the ordering found by our algorithm is

$$\begin{aligned} E(\sum_{(i,j) \in E} Y_{ij}^2) &= \sum_{(i,j) \in E} E(Y_{ij}^2) \\ &\leq \sum_{(i,j) \in E} O(|v_i - v_j|^2 \log^3 n) \\ &\leq O(OPT_{SDP}^2 \log^3 n) \\ &= O(OPT^2 \log^4 n), \end{aligned}$$

where OPT_{SDP} is the objective value of the SDP and hence (within a factor of $\sqrt{\log n}$) a lower bound on the minimum length of any linear ordering The result on the length of the ordering follows with high probability using Markov's inequality and taking square roots. $\square$

5 How good is the SDP?

What is the integrality gap of our first SDP? While our rounding procedure for the first SDP gives us an upper bound on the gap, it is possible that the gap is much smaller in reality. Note that our analysis is tight only for the *specific* rounding procedure we used, not the SDP itself.

Here we give some facts that indicate that the gap might be much smaller. One of the known lower bounds for the bandwidth of a graph is called the *density lower bound* [4]. It is defined as

$$B_d = \max_H \frac{|H| - 1}{\mathrm{diam}(H)},$$

where the maximum is taken over all connected subgraphs of G.

It is interesting to note that the density lower bound is approximately computable (within a factor of 2) in polynomial time as follows: Imagine picking a center node in the subgraph H^* achieving the bound, and consider the breadth-first tree rooted at this center truncated at level $\lceil \frac{\mathrm{diam}(H)}{2} \rceil$. The bound achieved by this subgraph is at least half the density lower bound. We can now search for the best such bound by looking over all choices of the root at all truncated breadth-first trees, for the best such subgraph.

The exact strength of the density lower bound is an open problem, but the largest known gap is $O(\log n)$ for an n-vertex graph. One of the known constructions of graphs which achieve this gap, the so-called Cantor combs, was described by Chung and Seymour [3].

The following lemma says that the integrality gap of our simple relaxation is no larger than the gap between the density lower bound and the optimum.

Lemma 10 *Let (x, b) be the optimal solution of the bandwidth SDP. Then $b = \Omega(B_d)$.*

Proof. Let H be the subgraph of G that achieves the maximum density. Since the average distance d_{ij} between points in the solution corresponding to vertices of H is $\Omega(|H|)$, there is a vertex v of H such that the total sum of distances between x_v and the other points in H is $\Omega(|H|^2)$. But this sum is at the same time at most $b|H|\mathrm{diam}(H)$, and so $b|H|\mathrm{diam}(H) = \Omega(|H|^2)$. That is,

$$b = \Omega\left(\frac{|H|}{\mathrm{diam}(H)}\right).$$

□

The results of Feige [6] imply that the optimum bandwidth is at most a poly-logarithmic factor higher than the density bound, thus implying a similar integrality gap for our SDP formulation of the bandwidth problem. This leaves open the tantalizing possibility of better rounding schemes of the SDP solution to the problem.

6 Conclusions and further work

Along the lines of the constraint set (9), and Feige's result [6], it is possible to refine the semi-definite relaxation further (by using the spreading constraints on k-simplices instead of just edges and triangles). This yields poly-logarithmic approximations for any L_{2k} norm in $O(n^{2k})$ time and also a poly-logarithmic approximation for minimum bandwidth in quasi-polynomial time ($n^{O(\log n)}$) by considering subsets of size $\log n$. It is an open question as to whether we can solve this latter relaxation in polynomial time.

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Appendix

Lemma 11 *If $u_1, \ldots, u_n$ are the points in the $(\lfloor \log n \rfloor + 1)$-dimensional space defined in the proof of lemma 8, there is a constant $c \geq 0.02$ such that $A(i,j,k) \geq c(j-i)\cdot(k-j)$ for all $i < j < k$.*

Proof. The idea of the proof is as follows: for a triangle defined by u_i, u_j and u_k, we consider its projection on a two-dimensional plane P_ℓ spanned by the coordinate vectors e_1 and e_ℓ, for different values of ℓ. Clearly, the area of each such projection is a lower bound on the area of the original triangle. The area of a triangle can be calculated as $\frac{1}{2}ab\sin\phi$ where a and b are two sides of the triangle and ϕ is the angle between them. If $u_i'u_j'u_k'$ is the projection of $u_iu_ju_k$ onto P_ℓ, then $|u_i' - u_j'| \geq (j-i)$ and $|u_j' - u_k'| \geq (k-j)$. Thus, if we can show that for each triple i, j, k there exists a coordinate ℓ such that the angle at u_j' (the projection of u_j onto P_ℓ) is bounded above by some universal constant ϕ, we will be done. In what follows we use an inductive case analysis to show that we can always ensure $\phi < 177.5°$ ($\sin\phi \geq 0.04$).

We assume without loss of generality that $j \leq n/2$ and $k > n/2$. (If $k \leq n/2$ or $i \geq n/2$ then we can work with $n/2$ instead of n and the claim holds by induction. The two cases, $j \geq n/2$ and $j \leq n/2$ are the same by symmetry so we only work with the first one.) If $k \geq 9n/16$ then after projecting to P_2, the angle at u_j' is at most $\sin^{-1}(1/8) < 173°$, so we can assume $n/2 < k < 9n/16$.

If $j \leq n/4$, then projecting onto P_3 works since the slope of the line through u_j' and u_k' is at most $1/16$, so the angle at u_j' is roughly $135°$ (precisely, the angle is at most $135° + \tan^{-1}(1/9) < 142°$). If $i \geq n/4$ the claim holds by induction.

Now there are four cases left.

1. If $i \leq n/8$ and $n/4 \leq j \leq 3n/8$, then the angle at u_j' in P_3 is at most $180° - \tan^{-1}(1/3) < 162°$.

2. If $n/8 \leq i \leq n/4$ and $n/4 \leq j \leq 3n/8$ we consider two subcases:

   (a) $(j - n/4) \leq 5/3(n/4 - i)$: Project onto P_3. The slope of the line through u_i' and u_j' is $\geq -1/4$ and the slope of the line through u_j' and u_k' is $\leq -1/3$. So, the angle at u_j' is at most $176°$.

   (b) $(j - n/4) \geq 5/3(n/4 - i)$: Project onto P_4. The slope of the line through u_i' and u_j' is $\geq 1/4$ and the slope of the line through u_j' and u_k' is $\leq 1/5$, so the angle at u_j' is at most $177.5°$.

3. If $i \leq n/8$ and $3n/8 \leq j \leq n/2$, then note that the slope in P_3 of the line between u'_i and u'_j is $\geq -1/3$. So, if the slope of the line through u'_j and u'_k in P_3 is $\leq -1/2$ then the angle at $u'_j \leq 172°$. Otherwise, it must be the case that the slope of the line through u'_j and u'_k in P_2 is $\leq 1/2$ (and the slope of the line through u'_i and u'_j in P_2 is 1), so in P_2 the angle at $u'_j \leq 162°$.

4. Finally, if $n/8 \leq i \leq n/4$ and $3n/8 \leq j \leq n/2$ then this is analogous to the previous case but using P_4 instead of P_3. In other words, in P_4, the slope of the line between u'_i and u'_j is $\geq -1/3$. So, if the slope of the line through u'_j and u'_k in P_4 is $\leq -1/2$ then the angle at $u'_j \leq 172°$. Otherwise the slope of the line in P_2 between u'_j and u'_k must be $\leq 1/2$ so in P_2 the angle at $u'_j \leq 162°$.

□