The Geometry of Logconcave Functions and Sampling Algorithms *

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Contents

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Abstract

The class of *logconcave* functions in $\mathbb{R}^n$ is a common generalization of Gaussians and of indicator functions of convex sets. Motivated by the problem of sampling from a logconcave density function, we study their geometry and introduce a technique for "smoothing" them out.

These results are applied to analyze two efficient algorithms for sampling from a logconcave distribution in n dimensions, with no assumptions on the local smoothness of the density function. Both algorithms, the ball walk and the hit-and-run walk, use a random walk (Markov chain) to generate a random point. After appropriate preprocessing, they produce a point from approximately the right distribution in time $O^*(n^4)$, and in amortized time $O^*(n^3)$ if many sample points are needed (where the asterisk indicates that dependence on the error parameter and factors of $\log n$ are not shown). These bounds match previous bounds for the special case when the distribution to sample from is the uniform distribution over a convex body.

1 Introduction

Virtually all known algorithms to sample from a high-dimensional convex body K (i.e., to generate a uniformly distributed random point in K) work by defining a Markov chain whose states are the points of K (or a sufficiently dense subset of it), and whose stationary distribution is uniform. Running the chain long enough produces an approximately uniformly distributed random point. The most thoroughly analyzed versions are the lattice walk [5, 8], the ball walk [14, 12, 11] and the hit-and-run walk [23, 24, 6, 15]. The hit-and-run walk, first proposed by Smith [23], has the same worst-case time complexity as the more thoroughly analyzed ball walk, but seems to be fastest in practice.

For what distributions is the random walk method efficient? These sampling algorithms can be extended to any other (reasonable) distribution in $\mathbb{R}^n$, but the methods for estimating their mixing time all depend on convexity properties. A natural class generalizing uniform distributions on convex sets is the class of *logconcave distributions*. For our purposes, it suffices to define these as probability distributions on $\mathbb{R}^n$ which have a density function f and the logarithm of f is concave. Well-known examples include the Boltzmann and Gaussian densities and the uniform density over a convex body. Such distributions play an important role in stochastic optimization [21] and other applications [10].

We assume that the function is given by an *oracle*, i.e., by a subroutine that returns the value of the function at any point x. We measure the complexity of the algorithm by the number of oracle calls.

The analysis of the lattice walk and ball walk have been extended to logconcave distributions [1, 7], but this analysis needs explicit assumptions on the Lipschitz constant of the distribution. In this paper, we avoid such assumptions by considering a smoother version of the function *in the analysis*. The smoother version is bounded by the original, continues to be logconcave, and has almost the same integral, i.e. it is almost equal at most points.

Our main result is that after appropriate preprocessing (bringing the distribution into isotropic position), both the ball walk (with a Metropolis filter) and the hit-and-run walk can be used to generate a sample using $O^*(n^4)$ oracle calls. We get a better bound of $O^*(n^3)$ if we consider a *warm start*. This means that we start the walk not from a given point but from a random point that is already quite well distributed in the sense that its density function is at most a constant factor larger than the target density f. While this sounds quite restrictive, it is often the case (for example, when generating many sample points, or using a "bootstrapping" scheme as in [12]) that this bound gives the actual cost of the algorithm per random point. Our amortized bound for sampling logconcave functions matches the best-known bound for the special case of sampling uniformly from a convex set. We also give an $O^*(n^5)$ algorithm for bringing an arbitrary logconcave distribution to isotropic position.

To justify the rather lengthy analysis of both algorithms, let us point out that on the one hand, the ball walk is simpler and more natural; on the other, the hit-and-run walk seems to be more efficient in practice and it has the important property that the dependence of its running time on the distance of the starting point from the boundary is only logarithmic (while this dependence is polynomial for the ball walk). This fact is proved in another paper [17].

Our analysis uses various geometric properties of logconcave functions; some of these are new, while others are well-known or folklore, but since a reference is not readily available, we prove them in Section 5.

2 Results

2.1 Preliminaries.

A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is *logconcave* if it satisfies

$$f(\alpha x + (1-\alpha)y) \geq f(x)^\alpha f(y)^{1-\alpha}$$

for every $x, y \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$. This is equivalent to saying that the support K of f is convex and $\log f$ is concave on K.

An integrable function $f : \mathbb{R}^n \to \mathbb{R}_+$ is a *density function*, if $\int_{\mathbb{R}^n} f(x)\,dx = 1$. Every non-negative integrable function f gives rise to a probability measure on the measurable subsets of $\mathbb{R}^n$ defined by

$$\pi_f(S) = \int_S f(x)\,dx \Big/ \int_{\mathbb{R}^n} f(x)\,dx\ .$$

The *centroid* of a density function $f:\mathbb{R}^n \to \mathbb{R}_+$ is the point

$$z_f = \int_{\mathbb{R}^n} f(x)x\,dx;$$

the *covariance matrix* of the function f is the matrix

$$V_f = \int_{\mathbb{R}^n} f(x)(x-z_f)(x-z_f)^T\,dx$$

(we assume that these integrals exist). The *variance* of the density function is

$$\mathsf{Var}(f) = \mathrm{tr}(V_f) = \int_{\mathbb{R}^n} |x-z_f|^2 f(x)\,dx.$$

For any logconcave function $f:\ \mathbb{R} \to \mathbb{R}^n$, we denote by M_f its maximum value. We denote by

$$L_f(t) = \{x \in \mathbb{R}^n:\ f(x) \geq t\}$$

its level sets, and by

$$f_t(x) = \begin{cases} f(x), & \text{if } f(x) \geq t, \\ 0, & \text{otherwise.} \end{cases}$$

its restriction to the level set. It is easy to see that f_t is logconcave.

A density function $f:\mathbb{R}^n \to \mathbb{R}_+$ is *isotropic*, if its centroid is 0, and its covariance matrix is the identity matrix. This latter condition can be expressed in terms of the coordinate functions as

$$\int_{\mathbb{R}^n} x_i x_j f(x)\,dx = \delta_{ij}$$

for all $1 \leq i,j \leq n$. This condition is equivalent to saying that for every vector $v \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} (v\cdot x)^2 f(x)\,dx = |v|^2.$$

In terms of the associated random variable X, this means that

$$\mathsf{E}(X) = 0 \quad \text{and} \quad \mathsf{E}(XX^T) = I.$$

For an isotropic density function in $\mathbb{R}^n$, we have $\mathsf{Var}(f) = n$.

We say that f is *near-isotropic up to a factor of* C, or shortly C*-isotropic*, if

$$\frac{1}{C} \leq \int (u^T x)^2\,d\pi_f(x) \leq C$$

for every unit vector u. The notions of "isotropic" and "non-isotropic" extend to non-negative integrable functions f, in which case we mean that the density function $f/\int_{\mathbb{R}^n} f$ is isotropic. Given any density function f with finite second moment $\int_{\mathbb{R}^n} |x|^2 f(x)\,dx$, there is an affine transformation of the space bringing it to isotropic position, and this transformation is unique except that it can be followed by an orthogonal transformation of the space.

We say that a density function is *a-rounded*, if for every $0 \le s \le 1$ it has a level set L of probability at most s that contains a ball of radius $a \cdot s$. In Section 5.4 (Lemma 5.13) we show that every isotropic logconcave density function is $(1/e)$-rounded.

Let f be a logconcave distribution in $\mathbb{R}^n$. For any line ℓ in $\mathbb{R}^n$, let $\mu_{\ell,f}$ be the measure induced by f on ℓ, i.e.

$$\mu_{\ell,f}(S) = \int_{p+tu\in S} f(p+tu)dt,$$

where p is any point on ℓ and u is a unit vector parallel to ℓ. We abbreviate $\mu_{\ell,f}(.)$ by $\mu_\ell(.)$ if f is understood, and also $\mu_\ell(\ell)$ by μ_ℓ. The probability measure $\pi_\ell(S) = \mu_\ell(S)/\mu_\ell$ is the *restriction* of f to ℓ.

For two points $u, v \in \mathbb{R}^n$, we denote by $d(u, v)$ their euclidean distance. For two probability distributions ν, τ on the same underlying σ-algebra, let

$$d_{\mathrm{tv}}(\nu, \tau) = \sup_A(\nu(A) - \tau(A))$$

be their total variation distance.

2.2 The random walks

Let f be a logconcave distribution on $\mathbb{R}^n$. We define two random walks on the points on $\mathbb{R}^n$.

The *ball walk* (with a Metropolis filter) to sample from f is defined as follows:

Ball walk.

- Pick a uniformly distributed random point y in the ball of radius r centered at the current point.
- Move to y with probability $\min(1, f(y)/f(x))$; stay at the current point with the remaining probability.

(The radius r will be specified shortly.)

The other random walk we study is the *hit-and-run walk*:

Hit-and-run walk.

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a random point y along the line ℓ chosen from the distribution π_ℓ.

A "uniformly distributed random line" through a point x means that it is uniformly distributed with respect to the measure we get from the uniform measure on the unit sphere by identify opposite points. We shall sometimes call this simply a *random line through* x.

It is easy to see that both walks are time-reversible. For technical reasons, we will also make them *lazy*, i.e., at each step with probability 1/2 we do nothing, and with probability 1/2 we do the above. We denote by P_u and Q_u the distributions obtained when we make a ball walk step and hit-and-run step from u, respectively.

Our first main theorem concerns functions that are near-isotropic (up to some fixed constant factor c). In Section 2.5, we discuss how to preprocess the function in order to achieve this.

Theorem 2.1 *If f is near-isotropic, then it can be approximately sampled in time $O^*(n^4)$ and in amortized time $O^*(n^3)$ if more than n sample points are needed; any logconcave function can be brought into near-isotropic position in time $O^*(n^5)$.*

Either the ball walk or the hit-run walk can be used in this algorithm. Theorem 2.1 is based on the following two more explicit results about a "warm start" for well-rounded density functions.

Theorem 2.2 *Let f be a logconcave density function in $\mathbb{R}^n$ that is a-rounded. Let σ be a starting distribution and assume that there is an $H > 0$ such that $\sigma(S) \leq H\pi_f(S)$ for every set S. Let σ^m be the distribution obtained after m steps of the ball walk applied to f. Then after m steps of the ball walk with*

$$r \leq \frac{a\varepsilon^2}{2^{10}H^2} \quad \text{and} \quad m > \frac{10^{10}n\mathsf{Var}(f)}{r^2}\log\frac{H}{\varepsilon},$$

the total variation distance between σ^m and π_f is less than ε.

For the hit-and-run walk, we prove:

Theorem 2.3 *Let f be a c-isotropic logconcave density function in $\mathbb{R}^n$. Let σ be a starting distribution and assume that there is an $H > 0$ such that $\sigma(S) \leq H\pi_f(S)$ for every set S. Let σ^m be the distribution obtained after m steps of the hit-and-run walk applied to f. Then for*

$$m > 10^{30}c^4H^4\frac{n^3}{\varepsilon^4}\ln^3\frac{2H}{\varepsilon},$$

the total variation distance of σ^m and π_f is less than ε.

In particular, we have

Corollary 2.4 *If f is near-isotropic, then with a warm start, both the ball walk and the hit-and-run walk mix in time $O^*(n^3)$.*

If we start from a single point, we may not get any bound on the mixing time at all: if the distribution has an unbounded support, and we start very far from the origin, then mixing may take arbitrarily long. However, if we start at the origin, then at least in the case of the hit-and-run walk, we only lose a factor of n [17]. It is an open question whether the results of [11] can be adopted to get rid of this additional factor of n (the "start penalty"). It is also open whether analogous result holds for the ball walk.

We end this section with a discussion of the implementation of these random walks. For both algorithms, the first step is easy to implement. For example, we can generate n independent random numbers $U_1, \ldots, U_n$ from the standard normal distribution, and consider the vector $U = (U_1, \ldots, U_n)$. For the ball walk, we generate a further random number η uniformly distributed in $[0, 1]$, and use vector $U' = (r\eta^{1/(n-1)}/\|U\|)U$ as the step. For the hit-end-run walk, we use the vector U to determine the direction of the line.

To describe the implementation of the second step, we have to discuss how the density function is given. We assume that it is given by an *oracle*: this means that for any $x \in \mathbb{R}^n$, the oracle returns the value $f(x)$. (We ignore here the issue that if the value of the function is irrational, the oracle only returns an approximation of f.) It would be enough to have an oracle which returns the value $C \cdot f(x)$ for some unknown constant $C > 0$ (this situation occurs in many sampling problems e.g., in statistical mechanics and simulated annealing).

For technical reasons, we also need a "guarantee" from the oracle that the centroid z_f of f satisfies $|z_f| \leq Z$ and that all the eigenvalues of the covariance matrix are bounded from above and below.

To implement the second step of the ball walk is trivial, but for the hit-and-run walk this needs a little thought. One way to do it is to use a binary search to find the point p on ℓ where the function is maximal, and the points a and b on both sides of p on ℓ where the value of the function is $\varepsilon_0 f(p)$. We allow a relative error of $\varepsilon_0 = \varepsilon/2$ (say), so the number of oracle calls is only $O(\log(1/\varepsilon))$. Then select a uniformly distributed random point y on the segment $[a, b]$, and independently a uniformly distributed random real number r in the interval $[0, 1]$. Accept y if $f(y) > rf(p)$; else, reject y and repeat. The distribution of the point generated this way is closer to the desired distribution than ε_0, and the expected number of oracle calls needed is $O(\log(1/\varepsilon))$. (For explicitly given density functions, there may be much more efficient direct methods.)

2.3 Distances

For two points $u, v \in \mathbb{R}^n$, let $\ell(u, v)$ denote the line through them. Let $[u, v]$ denote the segment connecting u and v, and let $\ell^+(u, v)$ denote the semiline in ℓ starting at u and not containing v. Furthermore, let

$$\begin{aligned} f^+(u, v) &= \mu_{\ell,f}(\ell^+(u, v)), \\ f^-(u, v) &= \mu_{\ell,f}(\ell^+(v, u)), \\ f(u, v) &= \mu_{\ell,f}([u, v]). \end{aligned}$$

We introduce the following "distance":

$$d_f(u,v) = \frac{f(u,v)f(\ell(u,v))}{f^-(u,v)f^+(u,v)}.$$

The function $d_f(u,v)$ does not satisfy the triangle inequality in general, but we could take $\ln(1 - d_f(u,v))$ instead, and this quantity would be a metric; however, it will be more convenient to work with d_f.

Suppose f is the uniform distribution over a convex set K. Let u, v be two points in K and p, q be the endpoints of $\ell(u,v) \cap K$, so that the points appear in the order p, u, v, q along $\ell(u,v)$. Then,

$$d_f(u,v) = d_K(u,v) = \frac{|u-v||p-q|}{|p-u||v-q|}.$$

2.4 Isoperimetry.

To bound the mixing time of our walk we use the conductance technique of Jerrum and Sinclair [9] and its extension to continuous Markov chains [16] (as do most of the previous papers). For the ball walk, for example, we show that any set of measure s has conductance $\Omega(as^2/nR)$ (see Section 9). The bound on the mixing time then follows from standard relationships between conductance and the mixing time. As we'll show, an isotropic density function is $(1/e)$-rounded and its variance is n, and hence the conductance of the ball walk in this case is $\Omega(\varepsilon^2/n^{3/2})$ and the mixing time is $O(n^3/\varepsilon^4)$.

As in almost all papers since [5], bounding the conductance depends on a geometric isoperimetric inequality. We will use an extension of Theorem 6 from [15] to logconcave functions.

Theorem 2.5 *Let f be a logconcave density function on $\mathbb{R}^n$ with support K. For any partition of K into three measurable sets S_1, S_2, S_3,*

$$\pi_f(S_3) \geq d_K(S_1, S_2)\pi_f(S_1)\pi_f(S_2).$$

To illustrate the theorem, it is useful to consider the situation where S_1 and S_2 are large subsets (in probability mass). Then, roughly speaking, the theorem says that the measure of the remainder grows with the d_K distance between S_1 and S_2.

2.5 Transforming to isotropic position.

The results of section 2.2 above use the assumption that the given density function is near-isotropic. To bring the function to this position can be considered as a preprocessing step, which only needs to be done once, and then we can generate any number of independent samples at the cost of $O^*(n^3)$ or $O^*(n^4)$ oracle calls per sample, as described earlier. But the problem of transforming into near-isotropic position is closely intertwined with the sampling problem, since we use sampling to transform an arbitrary logconcave density function into near-isotropic position.

In this section we describe an algorithm to achieve near-isotropic position. The following theorem is a consequence of a generalized Khinchine Inequality (Theorem 5.22) and Rudelson's Theorem [22] and is the basis of the algorithm.

Theorem 2.6 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an isotropic logconcave function. Let $v_1, v_2, \ldots, v_m$ be independent samples from π_f with*

$$m = 100 \cdot \frac{n}{\eta^2} \cdot \log^3 \frac{n}{\eta^2}.$$

Then

$$E\Big|\frac{1}{m}\sum_{i=1}^{m} v_i v_i^T - I\Big| \le \eta.$$

A corollary of this result is the following:

Corollary 2.7 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a (not necessarily isotropic) logconcave function. Let $v_1, v_2, \ldots, v_m$ be independent samples from π_f (where m is defined as above). Compute the matrices $V = \frac{1}{m}\sum_{i=1}^{m} v_i v_i^T$ and $W = V^{-1/2}$. Then with probability at least $1 - (\eta/\delta)$, we have*

$$|V - V_f| < \delta,$$

and hence the transformed function $\hat{f}(x) = f(Wx)$ is near-isotropic up to a factor of $1/(1-\delta)$.

The error probability of η/δ is not good enough for us; we cannot choose η small because the number of sample points m depends badly on η. The following trick reduces the error probability for a moderate cost:

Algorithm A.

- Choose $\eta = 1/100$ and $\delta = 1/10$.
- Repeat the construction in Corollary 2.7 q times, to get q matrices $V^{(1)}, \ldots, V^{(q)}$.
- For each of these matrices $V^{(i)}$, count how many other matrices $V^{(j)}$ satisfy

  $$|V^{(j)} - V^{(i)}|) < .2. \tag{1}$$

  If you find one for which this number is larger than $q/2$, return this as an approximation of V_f. Otherwise, the procedure fails.

Theorem 2.8 *With probability at least $1-(4/5)^q$, Algorithm A returns a matrix V satisfying $|V - V_f| < .3$.*

Proof. For each $1 \leq j \leq k$, we have

$$\mathsf{P}\Big(|V^{(j)} - V_f|) < .1\Big) > .9. \tag{2}$$

Hence (by Chernoff's inequality) with probability at least $1-(4/5)^k$, more than half of the $V^{(j)}$ satisfy (2). If both $V^{(i)}$ and $V^{(j)}$ satisfy (2), then they clearly satisfy (1), and hence in this case algorithm cannot fail. Furthermore, if $V^{(i)}$ is returned, then by pigeon hole, there is a j such that both (2) and (1) are satisfied, and hence $|V^{(i)} - V_f| < .3$ as claimed. □

Now the algorithm to bring f to near-isotropic position can be sketched as follows. Define

$$T_k = \frac{M_f}{2^{(1+1/n)^k}} \qquad k = 0, 1, \dots$$

The algorithm is iterative and in the k-th phase it brings the function $g_k = f_{T_k}$ into near-isotropic position.

Algorithm B.

- Choose $p = Cn \log n$ and $q = \log(p/\varepsilon)$.
- Bring the level set $\{x : \ f(x) \geq M_f/2\}$ to near-isotropic position (using e.g. the algorithm from [12]).
- For $k = 0, 1, \dots, p$, compute an approximation V of V_{g_k} using Algorithm A, and apply the linear transformation $V^{-1/2}$.

By Corollary 5.20, we have that after phase k, not only g_k is approximately isotropic, but also g_{k+1} is near-isotropic; in addition a random sample from g_k provides a warm start for sampling from g_{k+1} in the next phase. So we need to walk only $O^*(n^3)$ steps to get a point from (approximately) g_{k+1}. By Lemma 5.16 together with Theorem 5.14(e), the measure of the function outside the set $\{x : \ f(x) \geq M_f/T_{Cn \log n}\}$ is negligible and so $Cn \log n$ phases suffice to bring f itself to near-isotropic position.

The above description of the algorithm suits analysis, but is not how one implements it. We don't transform the body, rather, we transform the euclidean norm of the space. More precisely, we maintain a basis $u_1, \dots, u_n$, which is initialized as $u_i = e_i$ in phase 0. The u_i are fixed throughout each phase. To generate a line through the current point, we generate n independent random numbers $X_1, \dots, X_n$ from the standard Gaussian distribution, and compute the vector $X_1 u_1 + \dots X_n u_n$. The line we move on will be the line containing this vector.

At the end of phase k, we have generated m independent random points $v_1, \dots, v_m$ from the distribution $f_{T_{k+1}}$. We compute the matrix $W = \big(\frac{1}{m}\sum_{i=1}^m v_i v_i^T\big)^{1/2}$ and update the vectors u_i by letting $u_i = We_i$.

Each sample (after the first) takes $O^*(n^3)$ oracle calls and hence each phase takes $O^*(n^5)$ calls. The overall complexity of the algorithm is $O^*(n^5)$ oracle calls.

3 Outline of analysis

The proof of the main theorem about the mixing time is based on showing that the s-conductance [16] of the Markov chain is large. This is defined as

$$\phi_s = \inf_{A\subset\mathbb{R}^n, s<\pi_f(A)\leq 1/2} \frac{\Phi(A)}{\pi_f(A)-s},$$

where $\Phi(A)$ is the ergodic flow from A to its complement, $\bar{A}$. By the results of [16], specifically Corollary 1.6 (stated later in this paper as Theorem 9.1), a lower bound on ϕ_s directly gives a bound on the mixing time from a warm start, roughly about $1/\phi_s^2$.

The advantage of considering s-conductance is that we can ignore subsets that have measure smaller than s (the penalty is that we need some condition similar to a warm start). This fact allows us to replace the given density f (only in the analysis!) by another function $\hat{f} \leq f$, which is smoother, also logconcave and has almost the same integral.

To bound the s-conductance, we need two properties. First, if two points are "close," then the distributions obtained by taking one step from them have significant overlap. Here, the right notion of "close" is based on both Euclidean distance and the f-distance, d_f. This property is proven precisely as Lemma 7.1 for the ball walk and Lemma 7.2 for hit-and-run. Second, we need an isoperimetric inequality that puts a lower bound on the measure of points "near" the boundary of a subset. This is provided by Theorem 2.5.

To prove these properties and put them together in the final mixing proof, we will need several technical lemmas about spherical geometry and logconcave functions. We collect these in the next two sections.

4 Spheres and balls

We denote by π_n the volume of the unit ball B in $\mathbb{R}^n$. Our first lemma summarizes some folklore facts about volumes of sections of spheres.

Lemma 4.1 *Let H be a halfspace in $\mathbb{R}^n$ and B, a ball whose center is at a distance $t > 0$ from H. Then*

(a) *if $t \leq 1/\sqrt{n}$, then*

$$\text{vol}(H\cap B) > \left(\frac{1}{2} - \frac{t\sqrt{n}}{2}\right)\text{vol}(B);$$

(b) *if $t > 1/\sqrt{n}$ then*

$$\frac{1}{10t\sqrt{n}}(1-t^2)^{(n+1)/2}\text{vol}(B) \;<\; \text{vol}(H\cap B) \;<\; \frac{1}{t\sqrt{n}}(1-t^2)^{(n+1)/2}\text{vol}(B).$$

We define a function $t : [0, 1/2] \to \mathbb{R}$ that will be convenient to use in the sequel. Let C be a cap on the unit sphere S in $\mathbb{R}^n$, with radius r and let $\mathrm{vol}_{n-1}(C) = c\mathrm{vol}_{n-1}(S)$, $c < 1/2$. We define the function by

$$t(c) = \pi/2 - r.$$

Clearly $t(c)$ is a monotone decreasing function of c. This function is difficult to express exactly, but for our purposes, the following folklore bounds will be enough:

Lemma 4.2 (a) *If $0 < c < 2^{-n}$, then*

$$\frac{1}{2}c^{-1/n} < t(c) < 2c^{-1/n}.$$

(b) *If $2^{-n} < c < 1/4$, then*

$$\frac{1}{2}\sqrt{\frac{\ln(1/c)}{n}} < t(c) < 2\sqrt{\frac{\ln(1/c)}{n}};$$

(c) *If $1/4 < c < 1/2$, then*

$$\frac{1}{2}\left(\frac{1}{2} - c\right)\frac{1}{\sqrt{n}} < t(c) < 2\left(\frac{1}{2} - c\right)\frac{1}{\sqrt{n}}.$$

Using this function $t(c)$, we can formulate a fact that can be called "strong expansion" on the sphere:

Lemma 4.3 *Let T_1 and T_2 be two sets on the unit sphere S in $\mathbb{R}^n$, so that $\mathrm{vol}_{n-1}(T_i) = c_i\mathrm{vol}_{n-1}(S)$. Then the angular distance between T_1 and T_2 is at most $t(c_1) + t(c_2)$.*

Proof. Let d denote the angular distance between T_1 and T_2. The measure of T_i corresponds to the measure of a spherical cap with radius $\pi/2 - t(c_1)$. By spherical isoperimetry, the measure of the d-neighborhood of T_1 is at least as large as the measure of the d-neighborhood of the corresponding cap, which is a cap with radius $\pi/2 - t(c_1) + d$. The complementary cap has radius $\pi/2 + t(c_1) - d$ and volume at least c_2, and so it has radius at least $\pi/2 - t(c_2)$. Thus $\pi/2 + t(c_1) - d \geq \pi/2 - t(c_2)$, which proves the lemma. $\square$

The next lemma and its corollary quantify the following phenomenon: Consider a convex body K and imagine growing a sphere centered at some point inside K. For a while, the sphere is fully contained in K. When a significant portion of the sphere is outside K, it rapidly reaches a point where most of the sphere is outside K.

Lemma 4.4 *Let K be a convex body in $\mathbb{R}^n$ containing the unit ball B, and let $r > 1$. If $\phi(r)$ denotes the fraction of the sphere rS that is contained in K, then*

$$t(1 - \phi(r)) + t(\phi(2r)) \geq \frac{3}{8r}.$$

Proof. Let $T_1 = (rS) \setminus K$ and $T_2 = (1/2)((2rS) \cap K)$. We claim that the angular distance of T_1 and T_2 is at least $3/(8r)$. Consider any $y_1 \in T_1$ and $y_2 \in T_2$, we want to prove that the angle α between them is at least $3/(8r)$. We may assume that this angle is less than $\pi/4$ (else, we have nothing to prove). Let y_0 be the nearest point to 0 on the line through $2y_2$ and y_1. Then $y_0 \notin K$ by convexity, and so $s = |y_0| > 1$. Let α_i denote the angle between y_i and y_0. Then

$$\sin\alpha = \sin(\alpha_2 - \alpha_1) = \sin\alpha_2 \cos\alpha_1 - \sin\alpha_1 \cos\alpha_2.$$

Here, $\cos\alpha_1 = s/r$ and $\cos\alpha_2 = s/(2r)$; expressing the sines and substituting, we get

$$\sin\alpha = \frac{s}{r}\sqrt{1 - \frac{s^2}{4r^2}} - \frac{s}{2r}\sqrt{1 - \frac{s^2}{r^2}}.$$

We estimate this by standard tricks from below:

$$\begin{aligned}
\sin\alpha &= \frac{\frac{s^2}{r^2}\left(1 - \frac{s^2}{4r^2}\right) - \frac{s^2}{4r^2}\left(1 - \frac{s^2}{r^2}\right)}{\frac{s}{r}\sqrt{1 - \frac{s^2}{4r^2}} + \frac{s}{2r}\sqrt{1 - \frac{s^2}{r^2}}} > \frac{\frac{s^2}{r^2}\left(1 - \frac{s^2}{4r^2}\right) - \frac{s^2}{4r^2}\left(1 - \frac{s^2}{r^2}\right)}{\frac{s}{r} + \frac{s}{2r}} \\
&= \frac{s}{2r} > \frac{1}{2r}
\end{aligned}$$

Since $\alpha > \sin\alpha$, this proves the lemma. $\square$

The way this lemma is used is exemplified by the following:

Corollary 4.5 *Let K be a convex body in $\mathbb{R}^n$ containing the unit ball B, and let $1 < r < \sqrt{n}/32$. If K misses 1% of the sphere rS, then it misses at least 99% of the sphere $2rS$.*

5 The geometry of logconcave functions.

In this section we state geometric properties of logconcave functions that are used in the analysis of our algorithm. Many of these facts are well-known or even folklore, but references are not easy to pin down. Since the constants involved are needed to formalize our algorithm (not only to analyze it), we found that we have to include this section containing the proofs with explicit constants.

5.1 Marginals

We start with some definitions. The *marginal* of a function $f : \mathbb{R}^n \to \mathbb{R}_+$ on the set $S = \{i_1, \ldots, i_k\}$ of variables is defined by

$$G(x_{i_1}, \ldots, x_{i_k}) = \int_{\mathbb{R}^{n-k}} f(x_1, \ldots, x_n)\, dx_{j_1} \ldots dx_{j_{n-k}}, \tag{3}$$

where $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$. The *first marginal*

$$g(t) = \int_{x_2,\dots,x_n} f(t, x_2, \dots, x_n)\, dx_2 \dots dx_n$$

will be used most often. The *distribution function* of f is defined by

$$F(t_1, \dots, t_n) = \int_{x_1 \le t_1, \dots, x_n \le t_n} f(x_1, \dots, x_n)\, dx_1 \dots dx_n.$$

Clearly, the product and the minimum of logconcave functions is logconcave. The sum of logconcave functions is not logconcave in general; but the following fundamental properties of logconcave functions, proved by Dinghas [4], Leindler [18] and Prékopa [19, 20], can make up for this in many cases.

Theorem 5.1 *All marginals as well as the distribution function of a logconcave function are logconcave. The convolution of two logconcave functions is logconcave.*

We'll also need the following easy fact:

Lemma 5.2 *If f is in isotropic position, then so are its marginals.*

Proof. Let G be a marginal of f on the set $\{x_1, \dots, x_k\}$ of variables. Then for any two $1 \le i, j \le k$,

$$\begin{aligned}
&\int_{\mathbb{R}^k} x_i x_j G(x_1, \dots, x_k)\, dx_1 \dots dx_k \\
&\qquad = \int_{\mathbb{R}^k} x_i x_j \left(\int_{\mathbb{R}^{n-k}} f(x_1, \dots, x_n)\, dx_{k+1} \dots dx_n \right) dx_1 \dots dx_k \\
&\qquad = \int_{\mathbb{R}^n} x_i x_j f(x_1, \dots, x_n)\, dx_1 \dots dx_n = \delta_{ij}.
\end{aligned}$$

The proof of the fact that the centroid of the marginal is 0 is similar. □

5.2 One-dimensional functions

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function such that $g(x)$ tends to 0 faster than any polynomial as $x \to \infty$. Define its moments, as usual, by

$$M_n(f) = \int_0^\infty t^n g(t)\, dt.$$

Lemma 5.3 (a) *The sequence $(M_n(f) : \ n = 0, 1, \dots)$ is logconvex.*

(b) *If g is monotone decreasing, then the sequence defined by*

$$M'_n(f) = \begin{cases} nM_{n-1}(g)), & \textit{if } n > 0, \\ g(0) & \textit{if } n = 0. \end{cases}$$

is also logconvex.

(c) *If g is logconcave, then the sequence $M_n(g)/n!$ is logconcave.*

(d) *If g is logconcave, then*

$$g(0)M_1(g) \le M_0(g)^2.$$

(i.e., we could append $g(0)$ at the beginning of the sequence in (c) and maintain logconcavity).

Proof. (a) We have for every real x

$$0 \le \int_0^\infty (t+x)^2 t^n g(t)\,dt = x^2 M_n(g) + 2x M_g(n+1) + M_{n+2}(g).$$

Hence the discriminant of the quadratic polynomial on the right hand side must be non-positive:

$$M_{n+1}(g)^2 - M_n(g)M_{n+2}(g) \le 0,$$

which is just the logconvexity of the sequence.

(b) We may assume (by approximating) that g is differentiable. Then $-g'$ is nonnegative, which by (a) implies that the sequence $M_n(-g')$ is logconvex. Integrating by parts,

$$M_n(-g') = -\int_0^\infty t^n g'(t)\,dt = -[t^n g(t)]_0^\infty + \int_0^\infty n t^{n-1} g(t)\,dt = nM_{n-1}(g)$$

for $n \ge 1$, and

$$M_0(-g') = -\int_0^\infty g'(t)\,dt = g(0).$$

This proves (b).

(c) Let $h(t) = \beta e^{-\gamma t}$ be an exponential function ($\beta, \gamma > 0$) such that

$$M_n(h) = M_n(g) \qquad \text{and} \qquad M_{n+2}(h) = M_{n+2}(g)$$

(it is easy to see that such an exponential function exists). Then we have

$$\int_0^\infty t^n (h(t) - g(t))\,dt = 0, \qquad \int_0^\infty t^{n+2}(h(t) - g(t))\,dt = 0.$$

From this it follows that the graph of h must intersect the graph of g at least twice. By the logconcavity of g, the graphs intersect at exactly two points $a < b$. Furthermore, $h \le g$ in the interval $[a, b]$ and $h \ge g$ outside this interval. So the quadratic polynomial $(t-a)(t-b)$ has the same sign pattern as $h-g$, and hence

$$\int_0^\infty (t-a)(t-b)t^n (h(t) - g(t))\,dt \ge 0.$$

Expanding and rearranging, we get

$$\int_0^\infty t^{n+2}(h(t)-g(t))\,dt + ab\int_0^\infty t^n(h(t)-g(t))\,dt \ge (a+b)\int_0^\infty t^{n+1}(h(t)-g(t))\,dt.$$

The left hand side is 0 by the definition of h, so the right hand side is nonpositive, which implies that

$$M_{n-1}(h) = \int_0^\infty t^{n+1} h(t)\,dt \le \int_0^\infty t^{n+1} g(t)\,dt = M_{n+1}(g).$$

In other words, this shows that it is enough to verify the inequality for exponential functions. But for these functions, the inequality holds trivially.

(d) The proof is similar and is left to the reader. $\square$

Lemma 5.4 *Let X be a random point drawn from a one-dimensional logconcave distribution. Then*

$$\mathsf{P}(X \ge \mathsf{E}X) \ge \frac{1}{e}.$$

Proof. We may assume without loss of generality that $\mathsf{E}X = 0$. It will be convenient to assume that $|X| \le K$; the general case then follows by approximating a general logconcave distribution by such distributions.

Let $G(x) = \mathsf{P}(X \le x)$. Then G is logconcave, monotone increasing, and we have $G(x) = 0$ for $x \le -K$ and $G(x) = 1$ for $x \ge K$. The assumption that 0 is the centroid implies that

$$\int_{-K}^{K} xG'(x)\,dx = 0,$$

which by partial integration means that

$$\int_{-K}^{K} G(x)\,dx = K.$$

We want to prove that $G(0) \ge 1/e$.

The function $\ln G$ is concave, so it lies below its tangent at 0; this means that $G(x) \le G(0)e^{cx}$, where $c = G'(0)/G(0) > 0$. We may choose K large enough so that $1/c < K$. Then

$$G(x) \le \begin{cases} G(0)e^{cx} & \text{if } x \le 1/c, \\ 1 & \text{if } x > 1/c. \end{cases}$$

and so

$$\begin{aligned} K &= \int_{-K}^{K} G(x)\,dx \\ &\le \int_{-\infty}^{1/c} G(0)e^{cx}\,dx + \int_{1/c}^{K} 1\,dx \\ &= \frac{eG(0)}{c} + K - \frac{1}{c}, \end{aligned}$$

which implies that $G(0) \ge 1/e$ as claimed. $\square$

Lemma 5.5 *Let $g : \mathbb{R} \to \mathbb{R}_+$ be an isotropic logconcave density function.*
(a) *For all x, $g(x) \le 1$.*
(b) $g(0) \ge \frac{1}{8}$.

Proof. (a) Let the maximum of g be attained at a point z, and suppose that $g(z) > 1$. For $i = 0, 1, 2$, let

$$M_i = \int_z^\infty (x - z)^i \, g(x) \, dx,$$

$$N_i = \int_{-\infty}^z (z - x)^i \, g(x) \, dx.$$

Clearly

$$M_0 + N_0 = 1, \qquad N_1 - M_1 = z, \qquad M_2 + N_2 = 1 + z^2.$$

So

$$\begin{aligned} M_2 + N_2 &= (M_0 + N_0)^2 + (M_1 - N_1)^2 \\ &= (M_0 - M_1)^2 + (N_0 - N_1)^2 + 2(M_0 N_0 - M_1 N_1) + 2(M_0 M_1 + N_0 N_1) \\ &\ge 2(M_0 M_1 + N_0 N_1), \end{aligned}$$

since by Lemma 5.3(d), we have $M_1 \le M_0^2/g(z) \le M_0$, and similarly $N_1 \le N_0^2/g(z) \le N_0$.

On the other hand, by Lemma 5.3(c) and (d), we have

$$M_2 \le 2M_1^2/M_0 \le 2M_1 M_0/g(z) < 2M_1 M_0, \qquad N_2 < 2N_1 N_0,$$

and so

$$M_2 + N_2 < 2(M_0 M_1 + N_0 N_1),$$

a contradiction proving (a).

(b) We may assume that $g(x)$ is monotone decreasing for $x \ge 0$ (else, consider $g(-x)$). Let g_0 be the restriction of g to the nonnegative semiline. By Lemma 5.3(b),

$$M_1'(g_0)^3 \le M_0'(g_0) M_2'(g_0)^2,$$

which means that

$$M_0(g_0) \le g(0)^{2/3} (3M_2(g_0))^{1/3}.$$

Here trivially $M_2(g_0) \le M_2(g) = 1$, while Lemma 5.4 implies that

$$M_0(g_0) = \int_0^\infty g(t) \, dt \ge \frac{1}{e} \int_\infty^\infty g(t) \, dt = \frac{1}{e} M_0(g) = \frac{1}{e}.$$

Substituting these bounds, we get

$$g(0) \ge \sqrt{\frac{1}{3e^3}} \ge \frac{1}{8}.$$

$\square$

Lemma 5.5(a) is tight, as shown by the function

$$g(x) = \begin{cases} e^{-1-x}, & \text{if } x \geq -1, \\ 0, & \text{if } x < -1. \end{cases}$$

Part (b) is not tight; most probably the right constant is $1/(2 \cdot \sqrt{3})$, attained by the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$.

Lemma 5.6 *Let X be a random point drawn from a logconcave density function $g : \mathbb{R} \to \mathbb{R}_+$.*

(a) *For every $c \geq 0$,*

$$\mathsf{P}(g(X) \leq c) \leq \frac{c}{M_g}.$$

(b) *For every $0 \leq c \leq g(0)$,*

$$\mathsf{P}(\min g(2X), g(-2X) \leq c) \geq \frac{c}{4g(0)}.$$

Proof. (a) We may assume that the maximum of g is assumed at 0. Let $q > 0$ be defined by $g(q) = c$, and let $h(t) = g(0)e^{-\gamma t}$ be an exponential function such that $h(q) = g(q)$. Clearly such a γ exists, and $h(0) = g(0)$, $\gamma > 0$. By the logconcavity of the function, the graph of h is below the graph of g between 0 and q, and above outside. Hence

$$\frac{\int_q^\infty g(t)\,dt}{\int_0^\infty g(t)\,dt} \leq \frac{\int_q^\infty h(t)\,dt}{\int_0^\infty h(t)\,dt}$$

Here

$$\int_0^\infty h(t)\,dt = \frac{g(0)}{\gamma}, \int_q^\infty h(t)\,dt = \frac{g(q)}{\gamma} = \frac{c}{\gamma},$$

and so we get that

$$\int_q^\infty g(t)\,dt \leq \frac{c}{g(0)} \int_0^\infty g(t)\,dt.$$

Here $g(0) = M_g$, furthermore

$$\int_q^\infty g(t)\,dt = \mathsf{P}(X > q) = \mathsf{P}(g(X) < c, X > 0),$$

and

$$\int_0^\infty g(t)\,dt = \mathsf{P}(X > 0),$$

so we get that

$$\mathsf{P}(g(X) < c, X > 0) \leq \frac{c}{M_g}\mathsf{P}(X > 0).$$

Similarly,

$$\mathsf{P}(g(X) < c, X < 0) \le \frac{c}{M_g}\mathsf{P}(X < 0).$$

Adding up these two inequalities, the assertion follows.

(b) We may assume that $\mathsf{P}(X \le 0) \ge 1/2$. Let $q > 0$ be defined by $g(q) = c$. If $\mathsf{P}(X \ge q/2) > 1/4$ then the conclusion is obvious, so suppose that $\mathsf{P}(X \ge q/2) \le 1/4$. Similarly, we can assume that $\mathsf{P}(-q/2 < X < 0) \ge \frac{1}{4}$.

Let $h(t) = \beta e^{-\gamma t}$ be an exponential function such that

$$\int_{-q/2}^{0} h(t)\,dt = \int_{-q/2}^{0} g(t)\,dt = a,$$

and

$$\int_{q/2}^{q} h(t)\,dt = \int_{q/2}^{q} g(t)\,dt = b.$$

It is easy to see that such β and γ exist, and that $\beta, \gamma > 0$. From the definition of h we have

$$\begin{aligned} \frac{\beta}{\gamma}(e^{\gamma q/2} - 1) &= a, \\ \frac{\beta}{\gamma}(e^{-\gamma q/2} - e^{-\gamma q}) &= b. \end{aligned}$$

Dividing these equations with each other, we obtain

$$e^{\gamma q} = \frac{a}{b}.$$

By the logconcavity of the function, the graph of h must intersect the graph of g in a point in the interval $[-q/2, 0]$ as well as in a point in the interval $[q/2, q]$; it is below the graph of g between these two points and above outside. In particular, we get

$$c = g(q) \le h(q) = \beta e^{-\gamma q} = \beta\frac{b}{a},$$

and

$$g(0) \ge h(0) = \beta,$$

so

$$b \ge a\frac{c}{g(0)} \ge \frac{c}{4g(0)}.$$

□

We conclude with a useful lemma about the tail of a logconcave distribution.

Lemma 5.7 *Let X be a random point drawn from a logconcave distribution on $\mathbb{R}$. Assume that*

$$\mathsf{E}(X^2) \le 1.$$

Then for every $t > 1$,

$$\mathsf{P}(|X| > t) < e^{-t+1}.$$

Proof. First note that for $t \in [1,3]$ the bound follows using Chebychev's inequality: We have $\mathsf{E}(X^2) \geq t^2 \mathsf{P}(|X| > t)$, and so $\mathsf{P}(|X| > t) \leq 1/t^2$. From this, the conclusion follows if $t \leq 3$.

Fix $t > 3$ and let $f : \mathbb{R} \to \mathbb{R}_+$ be an integrable logconcave function satisfying

$$\int_{\mathbb{R}} x^2 f(x)\, dx \leq \int_{\mathbb{R}} f(x)\, dx. \tag{4}$$

and

$$\int_{-\infty}^{-t} f(x)\, dx + \int_t^{\infty} f(x)\, dx = Ce^{-t+1} \int_{\mathbb{R}} f(x)\, dx \tag{5}$$

where C is as large as possible. Our goal is to prove that $C < 1$.

Using a variant of Lemma 2.6 of [13], there exists an interval $[a,b] \in \mathbb{R}$ and a real γ satisfying

$$\int_a^b x^2 e^{\gamma x}\, dx \leq \int_a^b e^{\gamma x}\, dx \tag{6}$$

we have

$$\int_a^{-t} e^{\gamma x}\, dx + \int_t^b e^{\gamma x}\, dx = Ce^{-t+1} \int_a^b e^{\gamma x}\, dx. \tag{7}$$

We will prove that for any (a,b,γ) satisfying (6), we must have $C < 1$ in (7).

We can assume that $\gamma > 0$ by replacing (a,b,γ) by $(-b,-a,-\gamma)$ if necessary. Also, $b \geq 0$; if not, we can consider a logconcave function $\hat{f}$ which is zero outside $[a,0]$, nonzero in $[a,0]$, equal to $e^{\gamma x}$ for $x \in [a,-1]$ and

$$\int_{-1}^0 \hat{f}(x)\, dx = \int_{-1}^b e^{\gamma x}\, dx.$$

The function $\hat{f}$ has the same integral as $e^{\gamma x}$ overall and in the interval $[a,-t]$, but the second moment is smaller, i.e.,

$$\int_a^0 x^2 \hat{f}(x) < \int_a^b x^2 e^{\gamma x}\, dx.$$

So by rescaling $\hat{f}$ to make the moment larger without violating (4), we can obtain a function with a larger value of C. This contradicts the assumption about f.

We now evaluate both sides of (6) and cancel a common factor of $1/\gamma$ to get

$$e^{\gamma b}\left(b^2 - \frac{2b}{\gamma} + \frac{2}{\gamma^2}\right) - e^{\gamma a}\left(a^2 - \frac{2a}{\gamma} + \frac{2}{\gamma^2}\right) \leq e^{\gamma b} - e^{\gamma a}$$

which can be rewritten as

$$e^{\gamma b}\left(b^2 - \frac{2b}{\gamma} + \frac{2}{\gamma^2} - 1\right) \leq e^{\gamma a}\left(a^2 - \frac{2a}{\gamma} + \frac{2}{\gamma^2} - 1\right). \tag{8}$$

For any fixed $\gamma > 0$, the function

$$g(\gamma, x) = e^{\gamma x}\left(x^2 - \frac{2x}{\gamma} + \frac{2}{\gamma^2} - 1\right)$$

is monotone increasing in $(-\infty, -1)$, decreasing in $(-1, 1)$ and increasing again in $(1, \infty)$.

We next prove that $b < 2$. Note that $a \leq 1$ (otherwise, (6) is violated), and so $g(\gamma, a) \leq g(\gamma, -1)$. Further, (8) says that

$$g(\gamma, b) \leq g(\gamma, a), \tag{9}$$

which implies that $g(\gamma, b) \leq g(\gamma, -1)$. We get $b < 2$ using this and the inequality $g(\gamma, 2) > g(\gamma, -1)$ for $\gamma > 0$. The latter follows by verifying that

$$h(\gamma) = \gamma^2 e^{\gamma}(g(\gamma, 2) - g(\gamma, -1))$$

satisfies $h(0) = h'(0) = h''(0) = 0$ and $h''(\gamma) > 0$ for $\gamma > 0$.

Since $b < 2$ and $t > 3$, we can assume that $a < -3$. Again, from the inequality (9) and the monotonicity, we get $g(\gamma, a) \geq g(\gamma, 1)$. On the other hand, it can be verified by a routine calculation that $g(\gamma, a) < g(\gamma, 1)$ for $a = -1/(1-\gamma) < -3$ and since a is monotone increasing in this range, we must have $a \in [-1/(1-\gamma), -3]$. Thus, for $a \leq -3$, $\gamma \geq (a+1)/a$.

Using this, for $t \geq 3$,

$$\frac{\int_a^{-t} e^{\gamma x}\, dx}{\int_a^b e^{\gamma x}\, dx} \leq \frac{e^{-(a+1)t/a} - e^{a+1}}{1 - e^{a+1}} \leq e^{-(a+1)t/a} < e^{-t+1}$$

which proves the lemma. □

The lemma is tight (up to the +1 in the exponent) as shown by the function which is e^x for $x \leq 1$ and 0 otherwise.

5.3 Crossratios

For the next set of lemmas, it will be convenient to introduce the following notation: for a function $g: \mathbb{R} \to \mathbb{R}_+$ and $a < b$, let

$$g(a, b) = \int_a^b g(t)\, dt.$$

Furthermore, for $a < b < c < d$, we consider the cross-ratio

$$(a : c : b : d) = \frac{(d-a)(c-b)}{(b-a)(d-c)},$$

and its generalized version

$$(a : c : b : d)_g = \frac{g(a, d) g(b, c)}{g(a, b) g(c, d)}.$$

(The strange order of the parameters was chosen to conform with classical notation.) Clearly, $(a:c:b:d)_g = (a:c:b:d)$ if g is a constant function.

We start with a simple bound:

Lemma 5.8 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Then*

$$(a:c:b:d)_g \geq \frac{g(b)}{g(c)} - 1.$$

Proof. We may assume that $g(b) > g(c)$ (else, there is nothing to prove). Let $h(t)$ be an exponential function such that $h(b) = g(b)$ and $h(c) = g(c)$. By logconcavity, $g(x) \leq h(x)$ for $x \leq b$ and $g(x) \geq h(x)$ for $b \leq x \leq c$. Hence

$$\begin{aligned}
(a:c:b:d)_g &= \frac{g(a,d)g(b,c)}{g(a,b)g(c,d)} \geq \frac{g(b,c)}{g(a,b)} \\
&\geq \frac{h(b,c)}{h(a,b)} = \frac{h(c)-h(b)}{h(b)-h(a)} \geq \frac{h(c)-h(b)}{h(b)} \\
&= \frac{g(b)}{g(c)} - 1.
\end{aligned}$$

□

Lemma 5.9 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Then*

$$(a:c:b:d)_g \geq (a:c:b:d).$$

Proof. By Lemma 2.6 from [12], it suffices to prove this in the case when $g(t) = e^t$. Furthermore, we may assume that $a = 0$. Then the assertion is just Lemma 7 in [15]. □

Lemma 5.10 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c$. Then*

$$\frac{g(a,b)}{b-a} \leq \left(1 + \left|\ln \frac{g(b)}{g(c)}\right|_+\right) \frac{g(a,c)}{c-a}.$$

Proof. Let $h(t) = \beta e^{\gamma t}$ be an exponential function such that

$$\int_a^b h(t)\,dt = g(a,b) \qquad \text{and} \qquad \int_b^c h(t)\,dt = g(b,c).$$

It is easy to see that such β and γ exist, and that $\beta > 0$. The graph of h intersects the graph of g somewhere in the interval $[a,b]$, and similarly, somewhere in the interval $[b,c]$. By logconcavity, this implies that $h(b) \leq g(b)$ and $h(c) \geq g(c)$.

If $\gamma > 0$ then $h(t)$ is monotone increasing, and so

$$\frac{g(a,b)}{b-a} = \frac{h(a,b)}{b-a} \leq \frac{h(a,c)}{c-a} = \frac{g(a,c)}{c-a},$$

and so the assertion is trivial. So suppose that $\gamma < 0$. For notational convenience, we can rescale the function and the variable so that $\beta = 1$ and $\gamma = -1$. Also write $u = b - a$ and $v = c - b$. Then we have

$$g(a,b) = 1 - e^{-u} \qquad \text{and} \qquad g(a,c) = 1 - e^{-u-v}.$$

Hence

$$\frac{g(a,b)}{g(a,c)} = \frac{1-e^{-u}}{1-e^{-u-v}} \le \frac{u(v+1)}{u+v} = (v+1)\frac{b-a}{c-a}.$$

(The last step can be justified like this: $(1-e^{-u})/(1-e^{-u-v})$ is monotone increasing in u if we fix v, so replacing e^{-u} by $1-u < e^{-u}$ both in the numerator and denominator increases its value; similarly replacing e^{-v} by $1/(v+1)$ in the denominator decreases its value). To conclude, it suffices to note that

$$\ln\frac{g(b)}{g(c)} \ge \ln\frac{h(b)}{h(c)} = \ln\frac{e^{-u}}{e^{-u+v}} = v.$$

$\square$

The following lemma is a certain converse to Lemma 5.9:

Lemma 5.11 *Let $g:\ \mathbb{R} \to \mathbb{R}_+$ be a logconcave function and let $a < b < c < d$. Let $C = 1 + \max\{\ln(g(b)/g(a)), \ln(g(c)/g(d))\}$. If*

$$(a:c:b:d) \le \frac{1}{2C},$$

then

$$(a:c:b:d)_g \le 6C(a:c:b:d).$$

Proof. By the definition of $(a:c:b:d)$ and Lemma 5.10,

$$(a:c:b:d) = \frac{(d-a)(c-b)}{(b-a)(d-c)} > \frac{c-b}{b-a} > \frac{c-b}{c-a} \ge \frac{1}{C}\frac{g(b,c)}{g(a,c)}.$$

Hence by the assumption on $(a:c:b:d)$,

$$\frac{g(b,c)}{g(a,c)} = \frac{g(b,c)}{g(a,b)+g(b,c)} \le \frac{1}{2},$$

which implies that $g(a,b) \ge g(b,c)$. Similarly, $g(c,d) \ge g(b,c)$. We may assume by symmetry that $g(a,b) \le g(c,d)$. Then $g(a,d) = g(a,b) + g(b,c) + g(c,d) \le 3g(c,d)$, and so we have

$$(a:c:b:d)_g = \frac{g(a,d)g(b,c)}{g(a,b)g(c,d)} \le \frac{3g(b,c)}{g(a,b)} \le \frac{6g(b,c)}{g(a,c)}.$$

Using Lemma 5.10 again (for the order c, b, a), we get

$$(a:c:b:d)_g \le 6C\frac{c-b}{c-a} \le 6C\frac{c-b}{b-a} \le 6C\frac{(c-b)(d-a)}{(b-a)(d-c)} = 6C(a:b:c:d).$$

$\square$

5.4 Higher dimensional functions

Now consider a logconcave density function $f: \mathbb{R}^n \to \mathbb{R}$. Lemma 5.4 extends to any dimension without difficulty. A different proof for the special case when f is uniform over a convex body is given in [2].

Lemma 5.12 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a logconcave density function, and let H be any halfspace containing its centroid z. Then*

$$\int_H f(x)\, dx \geq \frac{1}{e}.$$

Proof. We may assume without loss of generality H is orthogonal to the first axis. Then the assertion follows by applying Lemma 5.4 to the first marginal of f. □

Lemma 5.13 *Every isotropic logconcave density function is $(1/e)$-rounded.*

Proof. We want to prove that if L is a level set of an isotropic logconcave density function f on $\mathbb{R}^n$, and L does not contain a ball of radius t, then $\int_L f(x)dx \leq et$. Let

$$h(x) = \begin{cases} f(x), & \text{if } x \in L, \\ 0, & \text{otherwise.} \end{cases}$$

Then h is logconcave. Let z be the centroid of h. Assume that L does not contain a ball of radius t. Then by the convexity of L, there exists $u \in \mathbb{R}^n$, $|u| = 1$ such that

$$\max_{x \in L} u^T(x - z) < t.$$

Rotate the coordinates so that $u = (1, 0, \ldots, 0)^T$, i.e.

$$\max_{x \in L} x_1 - z_1 < t.$$

The first marginal g of f is also logconcave. Further since f is in isotropic position, so is g. Lemma 5.5 implies that $g \leq 1$. Hence

$$\int_{x_1 \geq z_1} h(x)\, dx \leq \int_{z_1 \leq x_1 \leq z_1 + t} f(x)\, dx$$

$$= \int_{z_1 \leq x_1 \leq z_1 + t} g(x_1)\, dx_1 \leq t.$$

This bounds the probability of one "half" of L. But since h is a logconcave function, we have by Lemma 5.4 that

$$\int_L f(x)\, dx = \int_{\mathbb{R}^n} h(x)\, dx \leq et.$$

□

Theorem 5.14 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an isotropic logconcave density function.*

(a) *For every $v \in \mathbb{R}^n$ with $0 \le |v| \le 1/9$, we have $2^{-9n|v|} f(0) \le f(v) \le 2^{9n|v|} f(0)$.*

(b) *$f(x) \le 2^{2n+4} f(0)$ for every x.*

(c) *There is an $x \in \mathbb{R}^n$ such that $f(x) > (4e\pi)^{-n/2}$.*

(d) *$2^{-7n} \le f(0) \le n(20n)^{n/2}$.*

(e) *$f(x) \le 2^{8n} n^{n/2}$ for every x.*

Proof. (a) We prove the lower bound; the upper bound is analogous. Suppose that there is a point u with $|u| = t \le 1/9$ and $f(u) < 2^{-9nt} f(0)$. Let $v = (1/(9t))u$, then by logconcavity, $f(v) < 2^{-n} f(0)$. Let H be a hyperplane through v supporting the convex set $\{x \in \mathbb{R}^n : \ f(x) \ge f(v)\}$. We may assume that H is the hyperplane $x_1 = a$ for some $0 < a \le 1/9$. So $f(x) < 2^{-n} f(0)$ for every x with $x_1 = a$.

Let g be the first marginal of f. Then g is also isotropic, and hence $g(y) \le 1$ for all y, by Lemma 5.5. We also know by Lemma 5.4 that

$$\int_0^\infty g(y)\, dy \ge \frac{1}{e}.$$

We claim that

$$g(2a) \le \frac{g(a)}{2}. \tag{10}$$

Indeed, using the logconcavity of f, we get for every x with $x_1 = a$

$$f(2x) \le \frac{f(x)^2}{f(0)} \le \frac{f(x)}{2^n},$$

and hence

$$g(2a) = \int_{(x_1=2a)} f(x)\, dx_2 \dots dx_n \le 2^{-n} 2^{n-1} \int_{(x_1=a)} f(x)\, dx_2 \dots dx_n = \frac{g(a)}{2}.$$

Inequality (10) implies, by the logconcavity of g, that $g(x+a) \le g(x)/2$ for every $x \ge a$. Hence

$$\int_a^\infty g(y)\, dy \le 2 \int_a^{2a} g(y)\, dy \le 2a,$$

and hence

$$\int_0^\infty g(y)\, dy = \int_0^a g(y)\, dy + \int_a^\infty g(y)\, dy \le 3a < \frac{1}{e},$$

a contradiction.

(b) Let w be an arbitrary point with $f(w) > f(0)$. Let H be a hyperplane through 0 supporting the convex set $\{x \in \mathbb{R}^n : \ f(x) \ge f(0)\}$. We may assume that H is the hyperplane $x_1 = 0$. So $f(x) \le f(0)$ for every x with $x_1 = 0$. Let

g be the first marginal of f. Let H_t denote the hyperplane $x_1 = t$. We may assume that $w \in H_b$ with $b > 0$.

Let x be a point on H_0, and let x' be the intersection point of the line through w and x with the hyperplane $H_{b/2}$. Then by logconcavity,

$$f(w)f(x) \leq f(x')^2,$$

whence

$$f(x') \geq f(w)^{1/2} f(x)^{1/2} \geq \left(\frac{f(w)}{f(x)}\right)^{1/2} f(x) \geq \left(\frac{f(w)}{f(0)}\right)^{1/2} f(x),$$

and hence

$$g(b/2) = \int_{H_{b/2}} f(x)\,dx \geq 2^{1-n} \left(\frac{f(w)}{f(0)}\right)^{1/2} \int_{H_0} f(x)\,dx = 2^{1-n} \left(\frac{f(w)}{f(0)}\right)^{1/2} g(0).$$

So, using Lemma 5.5(b), we get

$$g(b/2) \geq 2^{-2-n} \left(\frac{f(w)}{f(0)}\right)^{1/2}.$$

On the other hand, by Lemma 5.5(a), we have $g(b/2) \leq 1$. This proves (b).

(c) For a random point X from the distribution, we have $\mathsf{E}(|X|^2) = n$, and hence by Markov's inequality, $\mathsf{P}(|X|^2 \leq 2n) \geq 1/2$. In other words,

$$\int_{\sqrt{2n}B} f(x)\,dx \geq \frac{1}{2}.$$

On the other hand,

$$\int_{\sqrt{2n}B} f(x)\,dx \leq M_f \mathrm{vol}(\sqrt{2n}B) = M_f (2n)^{n/2} \mathrm{vol}(B),$$

whence

$$M_f \geq \frac{1}{2(2n)^{n/2}\mathrm{vol}(B)} > (4\pi e)^{-n/2}.$$

(d) The lower bound follows from parts (b) and (c). Part (a) implies that

$$\int_{\mathbb{R}^n} f(x)\,dx \geq \int_{|x|\leq 1/9} f(x)\,dx \geq \frac{1}{9^n}\mathrm{vol}(B)\frac{f(0)}{2^n},$$

and since this integral is 1, we get that

$$f(0) \leq \frac{18^n}{\mathrm{vol}(B)} < n(20n)^{n/2}.$$

(e) is immediate from (d). $\square$

Lemma 5.15 *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an isotropic logconcave density function. Then for every line ℓ through 0,*

$$\int_\ell f(x)\,dx \geq 2^{-7n}.$$

(The integration is with respect to the Lebesgue measure on ℓ.)

Proof. We may assume that ℓ is the x_n-axis. Consider the marginal

$$h(x_1,\dots,x_{n-1}) = \int_{-\infty}^{\infty} f(x_1,\dots,x_{n-1},t)\,dt.$$

This is also an isotropic logconcave density function, so by Theorem 5.14(d) , we have

$$\int_\ell f(x)\,dx = h(0) \geq 2^{-7(n-1)} > 2^{-7n}.$$

□

The following lemma generalizes Lemma 5.6(a) to arbitrary dimension.

Lemma 5.16 *Let X be a random point drawn from a distribution with a logconcave density function $f : \mathbb{R}^n \to \mathbb{R}_+$. If $\beta \geq 2$, then*

$$\mathsf{P}(f(X) \leq e^{-\beta(n-1)} M_f) \leq (e^{1-\beta}\beta)^{n-1}.$$

Proof. We may assume that f is continuous, M_f is attained at the origin, and $M_f = f(0) = 1$. Let $c = e^{-\beta(n-1)}$. Using polar coordinates, we have

$$1 = \int_{\mathbb{R}^n} f(x)\,dx = \int_{S^{n-1}} \int_0^\infty f(tu)t^{n-1}\,dt\,du$$

and

$$\mathsf{P}(f(X) \leq c) = \int_{f(x)\leq c} f(x)\,dx = \int_{S^{n-1}} \int_{t:\, f(tu)\leq c} f(tu)t^{n-1}\,dt\,du$$

Fix a unit vector u, and let $q > 0$ be defined by $f(qu) = c$. By logconcavity,

$$f(tu) \begin{cases} \geq e^{-\beta(n-1)t/q}, & \text{if } t \leq q, \\ \leq e^{-\beta(n-1)t/q}, & \text{if } t \geq q. \end{cases}$$

Hence

$$\frac{\int_q^\infty f(tu)t^{n-1}\,dt}{\int_0^\infty f(tu)t^{n-1}\,dt} \leq \frac{\int_q^\infty e^{-\beta(n-1)t/q}t^{n-1}\,dt}{\int_0^\infty e^{-\beta(n-1)t/q}t^{n-1}\,dt}.$$

The integrals on the right hand side can be evaluated:

$$\int_0^\infty e^{-\beta(n-1)t/q}t^{n-1}\,dt = (n-1)!\left(\frac{\beta(n-1)}{q}\right)^{-n},$$

and

$$\int_q^\infty e^{-\beta(n-1)t/q} t^{n-1}\,dt = (n-1)!\left(\frac{\beta(n-1)}{q}\right)^{-n} e^{-\beta(n-1)} \sum_{k=0}^{n-1} \frac{(\beta(n-1))^k}{k!}.$$

The last sum can be estimated by $2(\beta(n-1))^{n-1}/(n-1)! < (e\beta)^{n-1}$. Thus

$$\int_q^\infty f(tu)t^{n-1}\,dt \le (e^{1-\beta}\beta)^{n-1} \int_0^\infty f(tu)t^{n-1}\,dt,$$

and so

$$\begin{aligned}
\mathsf{P}(f(X) \le c) &= \int_{u\in S}\int_{t\colon f(tu)\le c} f(tu)t^{n-1}\,dt \\
&\le (e^{1-\beta}\beta)^{n-1} \int_{u\in S}\int_0^\infty f(tu)t^{n-1}\,dt \\
&= (e^{1-\beta}\beta)^{n-1}.
\end{aligned}$$

$\square$

Lemma 5.17 *Let $X \in \mathbb{R}^n$ be a random point from a logconcave distribution with $\mathsf{E}(X^2) = C^2$. Then for any $R > 1$, $\mathsf{P}(|X| > RC) < e^{-R+1}$.*

Proof. We have

$$\int_{\mathbb{R}^n} (|x|^2 - C^2) f(x)\,dx = 0,$$

and if the assertion is false, then

$$\int_{|x|>RC} f(x)\,dx - e^{-R+1}\int_{\mathbb{R}^n} f(x)\,dx > 0.$$

The Localization Lemma (Corollary 2.4 in [13]), we have two points a, b and a $\gamma > 0$ so that

$$\int_0^1 (|(1-t)a + tb|^2 - C^2)(1+\gamma t)^n\,dt = 0, \tag{11}$$

and

$$\int\limits_{\substack{0\le t\le 1\\ |(1-t)a+tb|>RC}} (1+\gamma t)^n\,dt - e^{-R+1}\int\limits_0^1 (1+\gamma t)^n\,dt > 0. \tag{12}$$

It will be convenient to re-parametrize this segment $[a, b]$ by considering the closest point v of its line to the origin, and a unit vector u pointing in the direction of $b - a$. Let $R' = \sqrt{R^2C^2 - |v|^2}$, then we can rewrite (11) and (12) as

$$\int_{t_1}^{t_2} (|v|^2 + t^2 - C^2)(1 + \gamma'(t - t_1))^n\,dt = 0, \tag{13}$$

and

$$\int_{\substack{t_1 \le t \le t_2 \\ |t| > R'}} (1+\gamma'(t-t_1))^n \, dt - e^{-R+1} \int_{t_1}^{t_2} (1+\gamma'(t-t_1))^n \, dt > 0 \tag{14}$$

(with some t_1, t_2 and $\gamma > 0$). Equation (13) implies that $|v| < \sqrt{n}$. Introducing the new variable $s = t/\sqrt{C^2 - |v|^2}$, we get

$$\int_{s_1}^{s_2} (s^2-1)(1+\gamma''(s-s_1))^n \, ds = 0,$$

and

$$\int_{\substack{s_1 \le s \le s_2 \\ |s| > R''}} (1+\gamma''(s-s_1))^n \, ds - e^{-R+1} \int_{s_0}^{s_1} (1+\gamma''(s-s_1))^n \, ds > 0,$$

where $R'' = R'/\sqrt{C^2 - |v|^2} > R$, and s_1, s_2, γ'' are similarly transformed. Now this contradicts Lemma 5.7. $\square$

The following lemma generalizes the upper bound in Theorem 4.1 of [13]:

Lemma 5.18 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an isotropic logconcave function, and let z be a point where it assumes its maximum. Then $|z| \le n+1$.*

The characteristic function of an isotropic regular simplex shows that the bound is essentially tight.

Proof. Write $x = x + tu$, where $t \in \mathbb{R}_+$ and $|u| = 1$. Let $w = z/|z|$. We can write

$$1 = \int_{\mathbb{R}^n} (w^\mathsf{T} x)^2 f(x) \, dx = \int_{|u|=1} \int_0^\infty (w^\mathsf{T}(z+tu))^2 f(tu) t^{n-1} \, dt \, du.$$

Fix any u, and let $g(t) = f(tu)$. Then the inside integral is

$$\int_0^\infty (w^\mathsf{T}(z+tu))^2 g(t) t^{n-1} \, dt = |z|^2 M_{n-1}(g) + 2|z|(w^\mathsf{T}u) M_n(g) + (w^\mathsf{T}u)^2 M_{n+1}(g).$$

By Lemma 5.3(b), we have here

$$(n+1)^2 M_n(g)^2 \le n(n+2) M_{n-1}(g) M_{n+1}(g),$$

and so

$$|z|^2 \frac{n(n+2)}{(n+1)^2} M_{n-1}(g) + 2|z|(w^\mathsf{T}u) M_n(g) + (w^\mathsf{T}u)^2 M_{n+1}(g) \ge 0$$

(since the discriminant of this quadratic form is nonpositive). So

$$|z|^2 M_{n-1}(g) + 2|z|(w^\mathsf{T}u) M_n(g) + (w^\mathsf{T}u)^2 M_{n+1}(g) \ge \frac{1}{(n+1)^2} |z|^2 M_{n-1}(g).$$

Substituting this in the integral, we get

$$1 \geq \frac{1}{(n+1)^2}|z|^2,$$

which proves the lemma. $\square$

Lemma 5.19 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a logconcave function. Then for $0 < s < t < M_f$,*

$$\frac{\mathrm{vol}(L_f(s))}{\mathrm{vol}(L_f(t))} \leq \left(\frac{\ln(M_f/s)}{\ln(M_f/t)}\right)^n$$

Proof. Fix a point z where $f(z) = M_f$. Consider any point a on the boundary of $L(s)$. Let b be the intersection of the line through a and z with the boundary of $L(t)$. Let $b = ua + (1-u)z$. Since f is logconcave,

$$t \geq M_f^{1-u} s^u,$$

and so

$$u \geq \frac{\ln(M_f/t)}{\ln(M_f/s)}.$$

This means that $(\ln(M_f/s)/\ln(M_f/t))L(t)$ contains $L_f(s)$, and hence

$$\frac{\mathrm{vol}(L_f(s))}{\mathrm{vol}(L_f(t))} \leq \left(\frac{\ln(M_f/s)}{\ln(M_f/t)}\right)^n,$$

as claimed. $\square$

Lemma 5.20 *Let $f: \mathbb{R}^n \to \mathbb{R}$ be a logconcave function, and let f_t denote the restriction of f to the level set $L_f(t)$. Let $0 < s < t \leq M_f$ such that $t^{n+1} \leq s^n M_f$, and assume that f_t is isotropic. Then f_s is near-isotropic up to a factor of 6.*

Proof. Lemma 5.19 implies that

$$\int_{\mathbb{R}^n} f_s(x)\,dx \leq 3 \int_{\mathbb{R}^n} f_t(x)\,dx.$$

Hence for every unit vector u,

$$\begin{aligned}
\int_{\mathbb{R}^n} (u^T x)^2 \, d\pi_{f_s}(x) &= \frac{\int_{L_f(s)} (u^T x)^2 f(x)\,dx}{\int_{L_f(s)} f(x)\,dx} \\
&\geq \frac{\int_{L_f(t)} (u^T x)^2 f(x)\,dx}{\int_{L_f(s)} f(x)\,dx} \geq \frac{1}{3} \frac{\int_{L_f(t)} (u^T x)^2 f(x)\,dx}{\int_{L_f(t)} f(x)\,dx} \\
&= \frac{1}{3} \int_{\mathbb{R}^n} (u^T x)^2 \, d\pi_{f_t}(x).
\end{aligned}$$

On the other hand, Let L' be obtained by blowing up $L_f(t)$ from center z by a factor of $1+1/n$. Then $L_f(s) \subseteq L'$ by logconcavity, so

$$\begin{aligned}\int_{L_f(s)} (u^T x)^2 f(x)\,dx &\le \int_{L'} (u^T x)^2 f(x)\,dx \\ &= \Big(1+\frac{1}{n}\Big)^n \int_{L_f(t)} \Big[u^T\Big(\Big(1+\frac{1}{n}\Big)x - \frac{1}{n}z\Big)\Big]^2 f\Big(\Big(1+\frac{1}{n}\Big)x - \frac{1}{n}z\Big)\,dx.\end{aligned}$$

Using that f decreases along semilines starting from z, we get

$$\begin{aligned}&\int_{L_f(s)} (u^T x)^2 f(x)\,dx \\ &\qquad \le \Big(1+\frac{1}{n}\Big)^n \int_{L_f(t)} \Big[u^T\Big(\Big(1+\frac{1}{n}\Big)x - \frac{1}{n}z\Big)\Big]^2 f(x)\,dx\end{aligned}$$

We can expand this into three terms:

$$\begin{aligned}&\Big(1+\frac{1}{n}\Big)^{n+2} \int_{L_f(t)} (u^T x)^2 f(x)\,dx \\ &-2\frac{1}{n}\Big(1+\frac{1}{n}\Big)^{n+1} \int_{L_f(t)} (u^T x)(u^\mathsf{T} z) f(x)\,dx \\ &+\frac{1}{n^2}\Big(1+\frac{1}{n}\Big)^n \int_{L_f(t)} (u^\mathsf{T} z)^2 f(x)\,dx.\end{aligned}$$

Here the middle term is 0 since f_t is isotropic, and the last term is

$$\begin{aligned}&\frac{1}{n^2}\Big(1+\frac{1}{n}\Big)^n \int_{L_f(t)} (u^\mathsf{T} z)^2 f(x)\,dx \\ &\quad = \frac{1}{n^2}\Big(1+\frac{1}{n}\Big)^n (u^\mathsf{T} z)^2 \int_{L_f(t)} f(x)\,dx \\ &\quad < \frac{e}{n^2}|z|^2 \int_{L_f(t)} f(x)\,dx < 3\int_{L_f(t)} f(x)\,dx\end{aligned}$$

by Lemma 5.18. The first term is

$$\Big(1+\frac{1}{n}\Big)^{n+2} \int_{L_f(t)} (u^T x)^2 f(x))\,dx < 3\int_{L_f(t)} (u^T x)^2 f(x)\,dx,$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n} (u^T x)^2\, d\pi_{f_s}(x) &= \frac{\int_{L_f(s)} (u^T x)^2 f(x)\,dx}{\int_{L_f(s)} f(x)\,dx} \le 3 + 3\frac{\int_{L_f(t)} (u^T x)^2 f(x)\,dx}{\int_{L_f(t)} f(x)\,dx} \\ &= 3 + 3\int_{\mathbb{R}^n} (u^T x)^2\, d\pi_{f_t}(x) = 6.\end{aligned}$$

□

We end this section with an important folklore theorem, generalizing Khinchine's inequality to logconcave functions. First an elementary lemma, whose proof is omitted:

Lemma 5.21 *For every $\alpha \le \beta$,*

$$\frac{\int_\alpha^\beta e^{-t} t^k \, dt}{\int_\alpha^\beta e^{-t} \, dt} \le k^k \cdot \left(\frac{\int_\alpha^\beta e^{-t} |t| \, dt}{\int_\alpha^\beta e^{-t} \, dt} \right)^k$$

Using this lemma, we prove:

Theorem 5.22 *If X is a random point from a logconcave distribution in $\mathbb{R}^n$, then*

$$\mathsf{E}(|X|^k)^{1/k} \le 2k \mathsf{E}(|X|).$$

Note that the Hölder inequality gives an opposite relation:

$$\mathsf{E}(|X|^k)^{1/k} \ge \mathsf{E}(|X|).$$

Proof. We can write this inequality as

$$\frac{\int_{\mathbb{R}^n} f(x)|x|^k \, dx}{\int_{\mathbb{R}^n} f(x) \, dx} \le (2k)^k \cdot \left(\frac{\int_{\mathbb{R}^n} f(x)|x| \, dx}{\int_{\mathbb{R}^n} f(x) \, dx} \right)^k .$$

By Lemma 2.6 in [13], it suffices to prove that for any two points $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

$$\frac{\int_0^1 e^{ct} |ta + (1-t)b|^k \, dt}{\int_0^1 e^{ct} \, dt} \le (2k)^k \cdot \left(\frac{\int_0^1 e^{ct} |ta + (1-t)b| \, dt}{\int_0^1 e^{ct} \, dt} \right)^k .$$

Let c be the closest point of the segment $[a, b]$ to the origin. We can write any point on the segment $[a, b]$ as $c + s(b-a)$, where $\alpha = -|c-a|/|b-a| \le s \le \beta = |b-c|/|b-a|$. In this case,

$$|c + s(b-a)| \ge \max\{|c|, |s| \cdot |b-a|\}, \tag{15}$$

and of course

$$|c + s(b-a)| \le |c| + |s| \cdot |b-a|.$$

Hence

$$\begin{aligned} &\int_\alpha^\beta e^{c(s-\alpha)} |c + s(b-a)|^k \, ds \\ &\qquad \le 2^{k-1} \int_\alpha^\beta e^{c(s-\alpha)} |c|^k \, ds + 2^{k-1} \int_\alpha^\beta e^{c(s-\alpha)} |s|^k |b-a|^k \, ds \,. \end{aligned}$$

Here the first term is easy to evaluate, while the second term can be estimated using Lemma 5.21. We get that

$$\frac{\int_\alpha^\beta e^{c(s-\alpha)}|c+s(b-a)|^k\,ds}{\int_\alpha^\beta e^{c(s-\alpha)}\,ds} \le 2^{k-1}|c|^k + 2^{k-1}|b-a|^k k^k \left(\frac{\int_\alpha^\beta e^{c(s-\alpha)}|s|\,ds}{\int_\alpha^\beta e^{c(s-\alpha)}\,ds}\right)^k$$
$$= 2^{k-1}\left(\frac{\int_\alpha^\beta e^{c(s-\alpha)}|c|\,ds}{\int_\alpha^\beta e^{c(s-\alpha)}\,ds}\right)^k + 2^{k-1}k^k\left(\frac{\int_\alpha^\beta e^{c(s-\alpha)}|s||b-a|\,dt}{\int_\alpha^\beta e^{c(s-\alpha)}\,ds}\right)^k$$
$$\le (2k)^k\left(\frac{\int_\alpha^\beta e^{c(s-\alpha)}|c+s(b-a)|\,ds}{\int_\alpha^\beta e^{c(s-\alpha)}\,ds}\right)^k,$$

where the last step uses (15). □

6 Taming the function

6.1 Smoothing out

In this section, we define a "smoothed-out" version of the given density function f and prove its basic properties. Define

$$\hat f(x) = \inf_C \frac{1}{\mathrm{vol}(C)}\int_C f(x+u)\,du,$$

where C ranges over all convex subsets of the ball rB with $\mathrm{vol}(C) = \pi_n r^n/16$. (The parameter r will be specified later, and in the case of the ball walk it will be the radius of the ball used in each step.) Note that by the logconcavity of f, the level set $\{f \ge f(x)\}$ is convex, and hence there is a half-ball of B on which $f \le f(x)$. This implies that

$$\hat f(x) \le f(x). \tag{16}$$

The somewhat complicated definition of the function $\hat f$ serves to assure its logconcavity (Lemma 6.2). We'll also show that this function is not much smaller than f on the average (Lemma 6.3). We start with a simple observation that shows that we could (at the cost of a factor of 2) replace equality in the condition on C by inequality:

Lemma 6.1 *For every convex subset $D \subseteq rB$ with $\mathrm{vol}(D) \ge \mathrm{vol}(rB)/16$, we have*

$$\frac{1}{\mathrm{vol}(D)}\int_D f(x+u)\,du \ge \frac12 \hat f(x).$$

Proof. We can slice up D into convex sets $D = D_1 \cup \dots \cup D_m$ so that $\pi_n r^n/16 \le \mathrm{vol}(D_i) \le \pi_n r^n/8$. For at least one i, we have

$$\frac{1}{\mathrm{vol}(D_i)}\int_{D_i} f(x+u)\,du \le \frac{1}{\mathrm{vol}(D)}\int_D f(x+u)\,du.$$

Let C be any convex subset of D_i of volume $\mathrm{vol}(rD)/16$, then

$$\frac{1}{\mathrm{vol}(C)}\int_C f(x+u)\,du \le \frac{2}{\mathrm{vol}(D_i)}\int_{D_i} f(x+u)\,du.$$

Since by definition

$$\frac{1}{\mathrm{vol}(C)}\int_C f(x+u)\,du \ge \hat{f}(x),$$

this proves the lemma. □

Lemma 6.2 *The function $\hat{f}$ is logconcave.*

Proof. For a fixed convex set $C \subseteq rB$, the function

$$f_C(x) = \int_C f(x+u)\,du$$

is the convolution of the function f with the characteristic function of the convex set $-C$, and so it is logconcave by Theorem 5.1. Thus f is the infimum of the family $\{f_C\}$ of logconcave functions, and so it too is logconcave. □

Lemma 6.3 *Suppose f is a-rounded. Then*

$$\int_{\mathbb{R}^n} \hat{f}(x)\,dx \ge 1 - \frac{32}{a^{1/2}} r^{1/2} n^{1/4}.$$

If in particular f is isotropic, then

$$\int_{\mathbb{R}^n} \hat{f}(x)\,dx \ge 1 - 64 r^{1/2} n^{1/4}.$$

To prove Lemma 6.3, we need a lemma from [12] (in a paraphrased form).

Lemma 6.4 *Let K be a convex set containing a ball of radius t. Let X be a uniform random point in K and let Y be a uniform random in $X + sB$. Then*

$$\mathsf{P}(Y \notin K) \le \frac{s\sqrt{n}}{2t}.$$

Proof. [of Lemma 6.3]. Consider the set $\{(X,T) \in \mathbb{R}^n \times \mathbb{R}_+ :\ T < f(X)\}$, and select a pair (X,T) randomly and uniformly from this set. Choose a uniform random point Z in $X + rB$. We estimate the probability that $T > f(Z)$.

First, fix X and Z, and then choose T. Since T is uniform in the interval $[0, f(X)]$, the probability that $T > f(Z)$ is

$$\mathsf{P}_T(T > f(Z)) = \begin{cases} 0, & \text{if } f(Z) > f(X), \\ 1 - \frac{f(Z)}{f(X)}, & \text{if } f(Z) \le f(X). \end{cases}$$

which we can also write as

$$\mathsf{P}_T(T > f(Z)) = \max\left\{1 - \frac{f(Z)}{f(X)}, 0\right\}.$$

Taking the expectation also over the choice of Z,

$$\begin{aligned}\mathsf{P}_{T,Z}(T > f(Z)) &= \frac{1}{\pi_n r^n}\int_{X+rB} \mathsf{P}_T(T > f(z))\,dz \\ &= \frac{1}{\pi_n r^n}\int_{X+rB} \max\left\{1 - \frac{f(Z)}{f(X)}, 0\right\}\,dz.\end{aligned}$$

Let C be the convex set attaining the minimum in the definition of $\hat{f}(X)$, then it follows that

$$\begin{aligned}\mathsf{P}_{T,Z}(T > f(Z)) &\geq \frac{1}{\pi_n r^n}\int_C \left(1 - \frac{f(z)}{f(X)}\right)\,dz \\ &= \frac{1}{16}\left(1 - \frac{\hat{f}(X)}{f(X)}\right).\end{aligned}$$

Finally, taking expectation in X (which is from the distribution π_f), we get

$$\begin{aligned}\mathsf{P}_{T,Z,X}(T > f(Z)) &\geq \frac{1}{16}\int_{\mathbb{R}^n} \left(1 - \frac{\hat{f}(x)}{f(x)}\right) f(x)\,dx \\ &= \frac{1}{16}\left(1 - \int_{\mathbb{R}^n} \hat{f}(x)\,dx\right). \qquad (17)\end{aligned}$$

Next, start by choosing T from its appropriate marginal distribution, and then choose X uniformly from $L_f(T)$, and then choose Z. Fix some $t \geq f(Z)$ and let $c = \pi_f(L_f(t))$. We clearly have

$$\mathsf{P}(T > t) \leq \pi_f(L_f(t)) = c.$$

On the other hand, if $T \leq t$, then $L_f(T)$ contains a ball of radius ac, and so by Lemma 6.4, the probability that $Z \notin L_f(T)$ is at most $r\sqrt{n}/(ac)$. Hence

$$\mathsf{P}(t \geq T > f(Z)) \leq r\sqrt{n}/(ac).$$

Thus

$$\mathsf{P}(T > f(Z)) \leq c + r\sqrt{n}/(ac).$$

This bound is tightest if we choose t so that $c = r^{1/2}n^{1/4}/a^{1/2}$, in which case we get

$$\mathsf{P}(T > f(Z)) \leq 2\frac{r^{1/2}n^{1/4}}{a^{1/2}}.$$

We may not be able to choose this c, if $\pi_f(L_f(f(Z))) < c$; but then the inequality holds trivially.

Now comparing with (17), the lemma follows. $\square$

This lemma implies that the distributions π_f and $\pi_{\hat{f}}$ are close:

Corollary 6.5 *If f is isotropic, then the total variation distance between π_f and $\pi_{\hat{f}}$ is less than $80r^{1/2}n^{1/4}$.*

Proof. Let $c = 64r^{1/2}n^{1/4}$. The density function of $\pi_{\hat{f}}$ is at most $\hat{f}/(1-c)$, and hence (using that $\hat{f} \le f$ and $c < 1/5$),

$$\begin{aligned} d_{\mathrm{tv}}(\pi_f, \pi_{\hat{f}}) &= \int_{\mathbb{R}^n} \max\left\{\frac{\hat{f}(x)}{1-c} - f(x)\right\} dx \le \int_{\mathbb{R}^n} \frac{f(x)}{1-c} - f(x)\, dx = \frac{c}{1-c} \\ &< 80r^{1/2}n^{1/4}. \end{aligned}$$

□

Another simple consequence of Lemma 6.3 is the following:

Corollary 6.6 *Let X be a random point from an isotropic logconcave distribution with density function f. Then*

$$\mathsf{P}(\hat{f}(X) < \frac{1}{2}f(X)) \le 128r^{1/2}n^{1/4}.$$

Proof. Let

$$Z = \{x \in \mathbb{R}^n : \ \hat{f}(x) < \frac{1}{2}f(x)\}.$$

We have

$$\int_{\mathbb{R}^n} \hat{f}(x)\, dx \le \int_{\mathbb{R}^n \setminus Z} f(x)\, dx + \int_Z \frac{1}{2}f(x)\, dx = 1 - \frac{1}{2}\pi_f(Z).$$

By Lemma 6.3,

$$\pi_f(Z) \le 128r^{1/2}n^{1/4},$$

and the corollary follows. □

6.2 Smoothness measures

The quotient

$$\delta(x) = \frac{\hat{f}(x)}{f(x)}$$

is a certain measure of the smoothness of the function f at x. The value

$$\rho(x) = \frac{r}{16\sqrt{n}t(\delta(x)/4)} \approx \frac{r}{16\sqrt{\ln(4/\delta(x))}}$$

will also play an important role; the function is well behaved in a ball with radius $\rho(x)$ about x.

Lemma 6.7 *Let $x \in \mathbb{R}^n$, and let y be a uniform random point in $x+rB$. Then with probability at least* $15/16$,

$$f(y) \le \frac{2}{\delta(x)} f(x).$$

Proof. Consider the set

$$S = \left\{ u \in x + rB : \ f(u) > \frac{2}{\delta(x)} f(x) \right\}.$$

Clearly S is convex, and so is the set S' obtained by reflecting S in x. Furthermore, for every $y \in S'$ we have by logconcavity

$$f(y)f(2x-y) \le f(x)^2,$$

and since $f(2x-y) > 2f(x)^2/\hat{f}(x)$ by definition, we have $f(y) < \frac{1}{2}\hat{f}(x)$. By Lemma 6.1, this can only happen on a convex set of measure less than $1/16$, which proves the lemma. $\square$

Lemma 6.8 *For every $x, y \in \mathbb{R}^n$ with $|x-y| \le \frac{r}{2\sqrt{n}}$, we have*

$$\frac{\delta(x)}{2} \le \frac{f(y)}{f(x)} \le \frac{2}{\delta(x)}.$$

Proof. Let a be the closest point to x with $f(a) \le \hat{f}(x)/2$. Consider the supporting hyperplane of the convex set $\{y \in \mathbb{R}^n : \ f(y) \ge \hat{f}(x)/2\}$, and the open halfspace H bounded by this hyperplane that does not contain x. Clearly $f(y) < \hat{f}(x)$ for $y \in H$. By the definition of $\hat{f}$, it follows that the volume of the convex set $H \cap (x + rB)$ must be less than $\pi_n r^n/16$. On the other hand, by Lemma 4.1, the volume of this set is at least

$$\left(\frac{1}{2} - \frac{|a|\sqrt{n}}{2r} \right) \pi_n r^n.$$

Comparing these two bounds, it follows that

$$|a| > \frac{7}{8} \frac{r}{\sqrt{n}} > \frac{r}{2\sqrt{n}}.$$

This proves the first inequality. The second follows easily, since for the point $y' = 2x - y$ we have $|y'-x| = |y-x| < r/(2\sqrt{n})$, and so by the first inequality,

$$f(y') \ge \frac{\hat{f}(x)}{2}.$$

Then logconcavity implies that

$$f(y) \le \frac{f(x)^2}{f(y')} \le 2\frac{f(x)^2}{\hat{f}(x)},$$

as claimed. $\square$

Lemma 6.9 (a) *Let* $0 < q < (\delta(x)/4)^{1/n} r$. *Then,*

$$\mathrm{vol}\left(y \in x + qS \,:\, f(y) < \frac{\hat{f}(x)}{2}\right) \leq \left(1 - \frac{\delta(x)}{4}\right)\mathrm{vol}(qS).$$

(b) *Let* $0 < q \leq \rho(x)$. *Then,*

$$\mathrm{vol}\left(y \in x + qS \,:\, f(y) \leq \frac{\hat{f}(x)}{2}\right) \leq \frac{\delta(x)}{4}\mathrm{vol}(qS).$$

Proof. (a) We may assume, for notational convenience, that $x = 0$. Let $\delta = \delta(x)$ and $L = L(\hat{f}(0)/2)$. To prove (a), suppose that this fraction is larger than $1 - \delta/4$. Let $C = rB \cap H$, where H is a halfspace that avoids the points y with $f(y) > f(0)$ and also $\mathrm{vol}(H \cap (rB) = \pi_n r^n/16$ which. We can write

$$\int_C f(y)\,dy = \int_{C \setminus L} + \int_{C \cap L}.$$

Since $f(y) \leq f(0)$ for all $y \in C$ and $f(y) \leq \hat{f}(0)/2$ on the first set,

$$\int_C f(y)\,dy \leq \frac{\hat{f}(0)}{2}\mathrm{vol}(C \setminus L)) + f(0)\mathrm{vol}(C \cap L).$$

The first term can be estimated simply by $(\hat{f}(0)/2)\mathrm{vol}(C)$. The second term we split further:

$$\mathrm{vol}(C \cap L) \leq \mathrm{vol}((C \cap L) \setminus (qB)) + \mathrm{vol}(C \cap (qB)).$$

Since the fraction of every sphere tS, $t \geq q$, inside L is at most $\delta/4$, it follows that the first term is at most $\delta\mathrm{vol}(C)/4$. We claim that also the second term is less than $\delta\mathrm{vol}(C)/4$. Indeed,

$$\mathrm{vol}(C \cap L \cap (qB)) \leq \frac{1}{16}\mathrm{vol}(qB) = \frac{1}{16}\left(\frac{q}{r}\right)^n \mathrm{vol}(rB) \leq \frac{\delta}{4}\mathrm{vol}(C).$$

Thus

$$\int_C f(y)\,dy < \frac{\hat{f}(0)}{2}\mathrm{vol}(C) + 2f(0)\frac{\delta}{4}\mathrm{vol}(C) = \hat{f}(0)\mathrm{vol}(C),$$

which contradicts the definition of $\hat{f}$. This proves (a).

To prove (b), suppose that a fraction of more than δ of the sphere qS is not in L. On the other hand, a fraction of at least δ of the sphere $2qS$ is in L. This follows from part (a) if $q < \delta^{1/n} r$. If this is not the case, then we have

$$q \leq \rho(x) < \frac{r}{16\sqrt{\ln(4/\delta(x))}},$$

from where it is easy to conclude that $q < r/(2\sqrt{n})$. From Lemma 6.8 we get that all of the sphere qS is in L.

Now we apply Lemma 4.4 to the set L. By Lemma 6.8, this set contains a ball of radius $r/(2\sqrt{n})$ centered at zero. From the previous paragraph,

$$\frac{\mathrm{vol}(qS \setminus L)}{\mathrm{vol}(qS)} \geq \frac{\delta}{4} \quad \text{and} \quad \frac{\mathrm{vol}(2qS \cap L)}{\mathrm{vol}(qS)} \geq \frac{\delta}{4}.$$

Thus, after scaling so that the ball inside L has unit radius, Lemma 4.4 implies that

$$2t\left(\frac{\delta}{4}\right) \geq \frac{3r}{16q\sqrt{n}},$$

which contradicts the assumption that $q \leq \rho(x)$. □

For the hit-and-run walk, we need another smoothness measure, which (in view of Lemma 6.7, is analogous to $\delta(x)$. For a point $x \in K$, define $\alpha(x)$ as the smallest $s \geq 3$ for which

$$\mathsf{P}(f(y) \geq sf(x)) \leq \frac{1}{16},$$

where y is a hit-and-run step from x.

Lemma 6.10 *Let u be a random point from the stationary distribution π_f. For every $t > 0$,*

$$\mathsf{P}(\alpha(u) \geq t) \leq \frac{16}{t}.$$

Proof. If $t \leq 3$, then the assertion is trivial, so let $t \geq 3$. Then for every x with $a(x) \geq t$, we have

$$\mathsf{P}(f(y) \geq \alpha(x)f(x)) = \frac{1}{16},$$

and hence $\alpha(x) \geq t$ if and only if

$$\mathsf{P}(f(y) \geq tf(x)) \geq \frac{1}{16}.$$

Let $\mu(x)$ denote the probability on the left hand side. By Lemma 5.6(a), for any line ℓ, a random step along ℓ will go to a point x such that $f(x) \leq (1/t)\max_{y\in\ell} f(y)$ with probability at most $1/t$. Hence for every point u, the probability that a random step from u goes to a point x with $f(x) \leq (1/t)f(u)$ is again at most $1/t$. By time-reversibility, for the random point u we have

$$\mathsf{E}(\mu(u)) \leq \frac{1}{t}.$$

On the other hand,

$$\mathsf{E}(\mu(u)) \geq \frac{1}{16}\mathsf{P}\left(\mu(u) \geq \frac{1}{16}\right) = \frac{1}{16}\mathsf{P}(\alpha(u) \leq t),$$

which proves the lemma. $\square$

We study a hit-and-run step in greater detail. Let x be a point and ℓ, a line through x. We say that the pair (x, ℓ) is *ordinary*, if both points $u \in \ell$ with $|u - x| = \rho(x)$ satisfy $f(u) \geq \hat{f}(x)/2$. Note that this implies that $f(y) \geq \hat{f}(x)/2$ for all points y with $|y - x| \leq \rho(x)$.

Lemma 6.11 *Let $x \in \mathbb{R}^n$ and let ℓ be a random line through x. Then with probability at least $1 - \delta(x)/2$, (x, ℓ) is ordinary.*

Proof. If (x, ℓ) is not ordinary, then one of the points u on ℓ at distance $\rho(x)$ has $f(u) < f(x)/2$. By Lemma 6.9, the fraction of such points on the sphere $x + \rho(x)S$ is at most $\delta(x)/4$. So the probability that ℓ is not ordinary is at most $\delta(x)/2$. $\square$

Lemma 6.12 *Suppose that (x, ℓ) is ordinary. Let p, q be intersection points of ℓ with the boundary of $L(F/8)$ where F is the maximum value of f along ℓ, and let $s = \max\{\rho(x)/4, |x - p|/32, |x - q|/32\}$. Choose a random point y on ℓ from the distribution π_ℓ. Then*

$$\mathsf{P}(|x - y| > s) > \frac{\sqrt{\delta(x)}}{8}.$$

Proof. We may assume that $x = 0$. Suppose first that the maximum in the definition of s is attained by $\rho(0)/4$. Let y be a random step along ℓ, and apply lemma 5.6(b) with $c = \sqrt{f(0)\hat{f}(0)/2}$. We get that the probability that $f(2y) \leq c$ or $f(-2y) \leq c$ is at least

$$\frac{c}{4f(0)} = \frac{\sqrt{\delta(0)}}{8}.$$

Suppose $f(2y) \leq c$. Then logconcavity implies that $f(4y) \leq c^2/f(0) = \hat{f}(0)/4$. Since ℓ is ordinary, this means that in such a case $|4y| > \rho(x)$, and so $|x - y| = |y| > \rho(x)/4$.

So suppose that the maximum in the definition of s is attained by (say) $|p|/32$. We have the trivial estimates

$$\int_{|y|<s} f(y)\, dy \leq 2sF,$$

but

$$\int_\ell f(y)\, dy \geq |p - q| \frac{F}{8},$$

and so

$$\mathsf{P}(|y| \leq s) \leq \frac{16s}{|p - q|}.$$

Hence if $|p - q| > 24s$, then the conclusion of the lemma is valid.

So we may assume that $|p - q| < 24s$. Then q is between 0 and p, and so for every point y in the interval $[p, q]$, we have $|y| \geq 8s$. Since the probability of $y \in [p, q]$ is at least 7/8 the lemma follows again. $\square$

6.3 Cutting off small parts

The next two lemmas will allow us to ignore small troublesome regions that are very far from the center of gravity or where the function value is very small. We prove them for isotropic densities; the arguments are similar for nearly isotropic functions.

Lemma 6.13 *Let f be an isotropic density function in $\mathbb{R}^n$ with distribution π_f. For $t > 0$, let*

$$K_t = \{x \in \mathbb{R}^n : \ |x| \le t\sqrt{n}, \ f(x) > e^{-2(n-1)-2t} M_f\}.$$

Then

$$\pi_f(K_t) > 1 - 2e^{-t}.$$

Proof. Let $U = \{x \in \mathbb{R}^n : \ f(x) \le e^{-2(n-1)-2t} M_f\}$ and $V = \mathbb{R}^n \setminus t\sqrt{n}B$. Then by Lemma 5.16,

$$\pi_f(U) \le \left(\left(2 + \frac{2t}{n-1}\right) e^{-1-\frac{2t}{n-1}} \right)^{n-1} < e^{-t},$$

and by Lemma 5.17,

$$\pi_f(V) \le e^{-t},$$

and so

$$\pi_f(K_t) \ge 1 - \pi_f(U) - \pi_f(V) \ge 1 - 2e^{-t}.$$

□

Define

$$t_0 = 8 \ln \frac{2}{\varepsilon}, \quad R = t_0\sqrt{n},$$

and

$$K = K_{t_0} = \{x \in \mathbb{R}^n : \ |x| \le R, \ f(x) \ge e^{-2(n-1)-2R/\sqrt{n}} M_f\}.$$

For points in the interior of K, there are some simple but important relations between three distance functions we have to consider: the Euclidean distance $d(u, v)$, the f-distance $d_f(u, v)$ and the K-distance $d_K(u, v)$ (recall that this is the same as f-distance when f is replaced by the uniform distribution over K).

Lemma 6.14 *For any two points $u, v \in K$,*

(a) $d_K(u, v) \le d_f(u, v)$*;*

(b) $d_K(u, v) \ge \frac{1}{2R} d(u, v)$*;*

(c) $d_K(u, v) \ge \frac{1}{6n+6t_0} \min(1, d_f(u, v))$*.*

Proof. Part (a) follows from Lemma 5.9; (b) is immediate from the definition of K. For (c), we may suppose that $d_K(u, v) \le 1/(6n + 6t_0)$ (else, the assertion is obvious). By Lemma 5.16 and the definition of K, we have for any two points $x, y \in K$

$$\frac{f(x)}{f(y)} \le \frac{M_f}{f(y)} \le e^{2(n-1)+2t_0}.$$

So we can apply Lemma 5.11, with $C = 2n + 2t_0$ (for which $d_K(u,v) \le 1/2C$) and get that

$$d_K(u,v) \ge \frac{1}{6n+6t_0} d_f(u,v),$$

proving (c). $\square$

7 Geometric distance and probabilistic distance.

Our goal in this section is to show that if two points are close in a geometric sense, then the distributions obtained after making one step of the random walk (ball or hit-and-run) from them are also close in total variation distance. This will be relatively easy for the ball walk, but much more complicated for hit-and-run.

7.1 Ball walk

Recall that P_u be the distribution obtained on taking one ball step from u.

Lemma 7.1 *Let $u, v \in \mathbb{R}^n$ such that*

$$d(u,v) < \frac{r}{8\sqrt{n}} \qquad \text{and} \qquad d_f(u,v) < \frac{1}{8}.$$

Then

$$d_{\mathrm{tv}}(P_u, P_v) < 1 - \frac{\max\{\delta(u), \delta(v)\}}{18}$$

Proof. Let B_u, B_v be the balls of radius r around u and v, respectively. Suppose that $f(u) \le f(v)$, and let H be a halfspace with u on the boundary where $f(x) \le f(u)$. Let $C = B_u \cap B_v$ and $C' = H \cap C$. Since $d(u,v) \le r/(8\sqrt{n})$, we have

$$\mathrm{vol}(C') \ge \frac{1}{4}\mathrm{vol}(C) > \frac{1}{8}\mathrm{vol}(rB).$$

It follows by the definition of $\hat{f}$ that

$$\hat{f}(u), \hat{f}(v) \le \frac{2}{\mathrm{vol}(C')} \int_{C'} f(x)\,dx \le \frac{16}{\mathrm{vol}(rB)} \int_{C'} f(x)\,dx.$$

For any point $x \in C'$, the probability density of going from u to x or v to x is at least

$$\frac{1}{\mathrm{vol}(rB)} \frac{f(x)}{f(v)}.$$

Thus,

$$d_{\mathrm{tv}}(P_u, P_v) \le 1 - \frac{1}{\mathrm{vol}(rB)} \int_{x \in C'} \frac{f(x)}{f(v)} dx \le 1 - \frac{\max\{\hat{f}(u), \hat{f}(v)\}}{16 f(v)}.$$

Hence

$$d_{\mathrm{tv}}(P_u, P_v) \leq 1 - \frac{\hat{f}(v)}{16 f(v)} = 1 - \frac{1}{16}\delta(v).$$

By the condition that $\delta_f(u,v) < 1/8$ and by Lemma 5.8, we have

$$\frac{8}{9} \leq \frac{f(u)}{f(v)} \leq \frac{9}{8},$$

and so

$$d_{\mathrm{tv}}(P_u, P_v) \leq 1 - \frac{\hat{f}(u)}{16 f(u)} = 1 - \frac{1}{18}\delta(u).$$

□

7.2 Hit-and-run

Recall that Q_u be the distribution obtained on taking one ball step from u. It is not hard to see that

$$Q_u(A) = \frac{2}{n\pi_n} \int_A \frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}}. \tag{18}$$

The following lemma is the key to the analysis of the hit-and-run walk.

Lemma 7.2 *Let u, v be two points in $\mathbb{R}^n$ such that*

$$d_f(u,v) < \frac{1}{2^7 \ln(3+\alpha(u))} \quad \text{and} \quad d(u,v) < \frac{r}{2^{10}\sqrt{n}}.$$

Then

$$d_{tv}(Q_u, Q_v) < 1 - \frac{\delta(u)}{2^{12}}.$$

Proof. Let $\delta = \delta(u)$ and $\alpha = \alpha(u)$. We will show that there exists a set $A \subseteq K_0$ such that $Q_u(A) \geq \sqrt{\delta}/32$ and for every subset $A' \subset A$,

$$Q_v(A') \geq \frac{\sqrt{\delta}}{128} Q_u(A').$$

To this end, we define certain "bad" lines through u. Let σ be the uniform probability measure on lines through u. Let B_0 be the set of non-ordinary lines through u. By Lemma 6.11, $\sigma(B_0) \leq \delta/2$.

Let B_1 be the set of lines that are not almost orthogonal to $u - v$, in the sense that for any point $x \neq u$ on the line,

$$|(x-u)^T(u-v)| > \frac{2}{\sqrt{n}}|x-u||u-v|.$$

The measure of this subset can be bounded as $\sigma(B_1) \leq 1/8$.

Next, let B_2 be the set of all lines through u which contain a point y with $f(y) > 2\alpha f(u)$. By Lemma 5.6(a), if we select a line from B_2, then with probability at least 1/2, a random step along this line takes us to a point x with $f(x) \geq \alpha f(u)$. From the definition of α, this can happen with probability at most 1/16, which implies that $\sigma(B_2) \leq 1/8$.

Let A be the set of points in K which are not on any of the lines in $B_0 \cup B_1 \cup B_2$, and which are far from u in the sense of Lemma 6.12:

$$|x-u| \geq \max\left\{\frac{1}{4}\rho(u), \frac{1}{32}|u-p|, \frac{1}{32}|u-q|\right\}.$$

Applying Lemma 6.12 to each such line, we get

$$Q_u(A) \geq \left(1 - \frac{1}{8} - \frac{1}{8} - \frac{\delta}{2}\right)\frac{\sqrt{\delta}}{8} \geq \frac{\sqrt{\delta}}{32}.$$

We are going to prove that if we do a hit-and-run step from v, the density of stepping into x is not too small whenever $x \in A$. By the formula (18), we have to treat $|x-v|$ and $\mu_f(v,x)$.

We start by noting that $f(u)$ and $f(v)$ are almost equal. Indeed, Lemma 5.8 implies that

$$\frac{64}{65} \leq \frac{f(v)}{f(u)} \leq \frac{65}{64}.$$

Claim 1. For every $x \in A$,

$$|x-v|^n \leq \frac{2}{\sqrt{\delta}}|x-u|^n.$$

Indeed, since $x \in A$, we have

$$|x-u| \geq \frac{1}{4}\rho(u) = \frac{r}{64\sqrt{n}t(\delta/4)}$$

We can estimate this using Lemma 4.2. Assume that (b) applies (when (a) applies, the implication below follows easily). Then by the assumption of the present lemma,

$$\frac{r}{64\sqrt{n}t(\delta/4)} \geq \frac{r}{128\sqrt{\ln(4/\delta)}} \geq 8\frac{\sqrt{n}}{\sqrt{\ln(4/\delta)}}|u-v|.$$

On the other hand,

$$\begin{aligned}
|x-v|^2 &= |x-u|^2 + |u-v|^2 + 2(x-u)^T(u-v) \\
&\leq |x-u|^2 + |u-v|^2 + \frac{4}{\sqrt{n}}|x-u||u-v| \\
&\leq |x-u|^2 + \frac{\ln(4/\delta)}{64n}|x-u|^2 + \frac{\sqrt{\ln(4/\delta)}}{2n}|x-u|^2 \\
&\leq (1 + \frac{\ln(4/\delta)}{n})|x-u|^2
\end{aligned}$$

Hence:

$$|x-v|^n \le \left(1+\frac{\ln(4/\delta)}{n}\right)^{\frac{n}{2}} |x-u|^n < \frac{2}{\sqrt{\delta}}|x-u|^n.$$

The main part of the proof is the following claim:

Claim 2. For every $x \in A$,

$$\mu_f(v,x) < 64\frac{|x-v|}{|x-u|}\mu_f(u,x).$$

To prove this, let y, z be the points where $\ell(u,v)$ intersects the boundary of $L(f(u)/2)$, so that these points are in the order y, u, v, z. Let y', z' be the points where $\ell(u,v)$ intersects the boundary of K. By $f(y) = f(u)/2$, we have $f(y',u) \le 2f(y,u)$, and so (using logconcavity in the last step),

$$d_f(u,v) = \frac{f(u,v)f(y',z')}{f(y',u)f(v,z')} \ge \frac{f(u,v)}{f(y',u)} \ge \frac{f(u,v)}{2f(y,u)} \ge \frac{|u-v|}{4|y-u|}.$$

It follows that

$$|y-u| \ge \frac{|u-v|}{4d_f(u,v)} \ge 32\ln(3+\alpha)\cdot|u-v| > 32|u-v|. \tag{19}$$

A similar argument shows that

$$|z-v| \ge 32\ln(3+\alpha)\cdot|u-v| > 32|u-v|. \tag{20}$$

Next, we compare the function values along the lines $\ell(u,x)$ and $\ell(v,x)$. Let F denote the maximum value of f along $\ell(u,x)$, and let p, q be the intersection points of $\ell(u,x)$ with the boundary of $L(F/8)$, so that q is in the same direction from p as x is from u. Since $x \in A$, we know that

$$|u-p|, |u-q| \le 32|x-u|. \tag{21}$$

For each point $a \in \ell(u,x)$ we define two points $a', a'' \in \ell(v,x)$ as follows. If a is on the semiline of $\ell(u,x)$ starting from x containing u, then we obtain a' by projecting a from y to $\ell(v,x)$, and we obtain a'' by projecting a from z. If a is on the complementary semiline, then the other way around, we obtain a' by projecting from z and a'' by projecting from y.

Simple geometry shows that if

$$|a-u| < \frac{|y-u|}{|u-v|}|x-u|, \frac{|z-u|}{|u-v|}|x-u|,$$

then a', a'' exist and a'' is between v and a'. Furthermore, $a \mapsto a'$ and $a \mapsto a''$ are monotone mappings in this range.

A key observation is that if $|a-u| \le 32|x-u|$, then

$$f(a') < 2f(a). \tag{22}$$

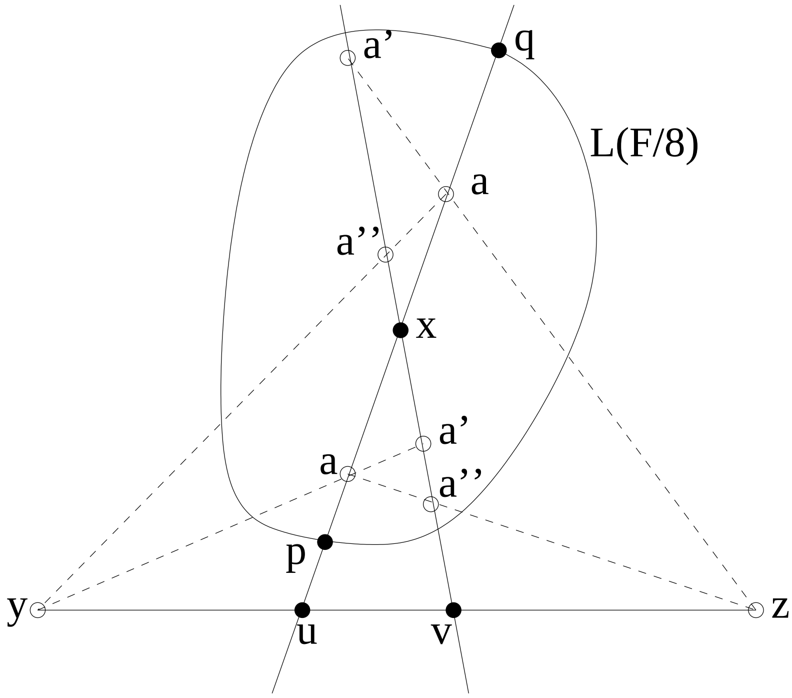

Figure 1: Comparing steps from nearby points.

To prove this, let $b = a'$. We have to distinguish three cases.

(a) $a \in [u, x]$. Then, using (19),

$$\frac{|a-b|}{|y-b|} \leq \frac{|u-v|}{|y-v|} \leq \frac{|u-v|}{|y-u|} \leq \frac{1}{128 \ln(3+\alpha)}.$$

Further, by the logconcavity of f,

$$f(a) \geq f(b)^{\frac{|y-a|}{|y-b|}} f(y)^{\frac{|a-b|}{|y-b|}}.$$

Thus,

$$f(b) \leq \frac{f(a)^{\frac{|y-b|}{|y-a|}}}{f(y)^{\frac{|a-b|}{|y-a|}}} \quad = \quad f(a) \left(\frac{f(a)}{f(y)}\right)^{\frac{|a-b|}{|y-a|}}.$$

Here

$$f(a) \leq 2\alpha f(u) \leq 4\alpha f(y),$$

since $x \in A_2$. Thus

$$f(b) \leq f(a)(4\alpha)^{\frac{1}{128 \ln(3+\alpha)}} < 2f(a).$$

(b) $a \in \ell^+(x,u)$. By Menelaus' theorem,

$$\frac{|a-b|}{|b-z|} = \frac{|x-a|}{|x-u|} \cdot \frac{|u-v|}{|v-z|}.$$

By (21), $|x-a|/|x-u| \le 16$, and so by (20),

$$\frac{|a-b|}{|b-z|} \le 16 d_f(u,v) \le \frac{1}{4\ln(3+\alpha)}.$$

By logconcavity,

$$f(a) \ge f(b)^{|a-z|/|b-z|} f(z)^{|a-b|/|b-z|}$$

Rewriting, we get

$$\begin{aligned} f(b) \le \frac{f(a)^{|b-z|/|a-z|}}{f(z)^{|a-b|/|a-z|}} &= f(a)\left(\frac{f(a)}{f(z)}\right)^{|a-b|/|a-z|} \\ &\le f(a)(4\alpha)^{\frac{1}{4\ln(3+\alpha)-1}} \le 2f(a). \end{aligned}$$

(c) $a \in \ell^+(u,x)$. By Menelaus' theorem again,

$$\frac{|a-b|}{|b-y|} = \frac{|x-a|}{|x-u|} \cdot \frac{|u-v|}{|v-y|}.$$

Again by (21), $|x-a|/|x-u| \le 16$. Hence, using (19) again,

$$\frac{|a-b|}{|b-z|} \le 16 d_f(u,v) \le \frac{1}{4\ln(3+\alpha)}.$$

By logconcavity,

$$f(a) \ge f(b)^{|a-y|/|b-y|} f(z)^{|a-b|/|b-y|}$$

Rewriting, we get

$$\begin{aligned} f(b) \le \frac{f(a)^{|b-y|/|a-y|}}{f(y)^{|a-b|/|a-y|}} &= f(a)\left(\frac{f(a)}{f(y)}\right)^{|a-b|/|a-y|} \\ &\le f(a)(4\alpha)^{\frac{1}{4\ln(3+\alpha)-1}} \le 2f(a). \end{aligned}$$

This proves inequality (22).

Similar argument shows that if $|a-u| \le 32|x-u|$, then

$$f(a'') > \frac{1}{2} f(a). \tag{23}$$

Let $a \in \ell(u,x)$ be a point with $f(a) = F$. Then $a \in [p,q]$, and hence $|a-u| < \max\{|p-u|, |q-u|\} \le 32|x-u|$ (since $x \in A$).

These considerations describe the behavior of f along $\ell(v,x)$ quite well. Let $r = p'$ and $s = q'$. (22) implies that $f(r), f(r) \le F/4$. On the other hand, $f(a'') > F/2$ by (23).

Next we argue that $a'' \in [r,s]$. To this end, consider also the point $b \in \ell(u,x)$ defined by $b' = a''$. It is easy to see that such a b exists and that b is between u and a. This implies that $|b-u| < 32|x-u|$, and so by (22), $f(b) > f(b')/2 = f(a'')/2$. Thus $f(b) > F/4$, which implies that $b \in [p,q]$, and so $b' \in [p',q'] = [r,s]$.

Thus f assumes a value at least $F/2$ in the interval $[r,s]$ and drops to at most $F/4$ at the ends. Let c be the point where f attains its maximum along the line $\ell(v,x)$. It follows that $c \in [r,s]$ and so $c = d'$ for some $d \in [p,q]$. Hence by (22), $f(c) \le 2f(b) \le 2F$. Thus we know that the maximum value F' of f along $\ell(v,x)$ satisfies

$$\frac{1}{2}F \le F' \le 2F. \tag{24}$$

Having dealt with the function values, we also need an estimate of the length of $[r,s]$:

$$|r-s| \le 2\frac{|x-v|}{|x-u|}|p-q|. \tag{25}$$

To prove this, assume e.g. that the order of the points along $\ell(u,x)$ is p,u,x,q (the other cases are similar). By Menelaus' theorem,

$$\frac{|x-r|}{|v-r|} = \frac{|u-y|}{|v-y|} \cdot \frac{|x-p|}{|u-p|} = \left(1 - \frac{|v-u|}{|v-y|}\right)\frac{|x-p|}{|u-p|}.$$

Using (19), it follows that

$$\frac{|x-r|}{|v-r|} \ge \frac{31}{32}\frac{|x-p|}{|u-p|}.$$

Thus,

$$\begin{aligned}
\frac{|x-v|}{|v-r|} &= \frac{|x-r|}{|v-r|} - 1 \ge \frac{31}{32}\frac{|x-p|}{|u-p|} - 1 \\
&= \frac{|x-u|}{|u-p|} - \frac{1}{32}\frac{|x-p|}{|u-p|} \\
&= \frac{|x-u|}{|u-p|}\left(1 - \frac{1}{32}\frac{|x-p|}{|x-u|}\right) \\
&> \frac{|x-u|}{|u-p|}\left(1 - \frac{1}{32}\cdot 16\right) = \frac{1}{2}\frac{|x-u|}{|u-p|}.
\end{aligned}$$

In the last line above, we have used (21). Hence,

$$|v-r| < 2\frac{|x-v|}{|x-u|}|u-p|. \tag{26}$$

Similarly,

$$|v-s| < 2\frac{|x-v|}{|x-u|}|u-q|.$$

Adding these two inequalities proves (25).

Now Claim 2 follows easily. We have

$$\mu(\ell(u,x)) \geq \frac{F}{8}|p-q|, \tag{27}$$

while we know by Lemma 5.6(a) that

$$\mu(\ell(v,x)) \leq 2f[r,s].$$

By (24) and (25),

$$f(r,s) \leq 2F|r-s| \leq 4F|p-q|\frac{|x-v|}{|x-u|},$$

and hence, using (27),

$$\mu(\ell(v,x)) < 64\frac{|x-v|}{|x-u|}\mu(\ell(u,x)),$$

proving Claim 2.

Using Claims 1 and 2, we get for any $A' \subset A$,

$$\begin{aligned} Q_v(A') &= \frac{2}{n\pi_n}\int_{A'}\frac{f(x)\,dx}{\mu_f(v,x)|x-v|^{n-1}} \\ &\geq \frac{2}{64n\pi_n}\int_{A'}\frac{|x-u|f(x)\,dx}{\mu_f(u,x)|x-v|^n} \\ &\geq \frac{2\sqrt{\delta}}{128n\pi_n}\int_{A'}\frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}} \\ &\geq \frac{\sqrt{\delta}}{128}Q_u(A'). \end{aligned}$$

This concludes the proof of Lemma 7.2. □

8 Proof of the isoperimetric inequality.

Here we prove Theorem 2.5. Let h_i be the characteristic function of S_i for $i = 1, 2, 3$, and let h_4 be the constant function 1 on K. We want to prove that

$$d_K(S_1,S_2)\left(\int fh_1\right)\left(\int fh_2\right) \leq \left(\int fh_3\right)\left(\int fh_4\right).$$

Let $a, b \in K$ and g be a nonnegative linear function on $[0,1]$. Set $v(t) = (1-t)a + tb$, and

$$J_i = \int_0^1 h_i(v(t))f(v(t))g^{n-1}(v(t))\,dt.$$

By Theorem 2.7 of [13], it is enough to prove that

$$d_K(S_1, S_2) J_1 \cdot J_2 \leq J_3 \cdot J_4. \tag{28}$$

A standard argument [12, 15] shows that it suffices to prove the inequality for the case when J_1, J_2, J_3 are integrals over the intervals $[0, u_1]$, $[u_2, 1]$ and (u_1, u_2) respectively $(0 < u_1 < u_2 < 1)$.

Consider the points $c_i = (1 - u_i)a + u_i b$. Since $c_i \in S_i$, it is easy to see that

$$d_K(c_1, c_2) \leq (a : u_2 : u_1 : b)$$

while

$$\frac{J_3 \cdot J_4}{J_1 \cdot J_2} = (a : u_2 : u_1 : b)_f.$$

Thus (28) follows from Lemma 5.9.

9 Proof of the mixing bounds.

Consider a random walk on $\mathbb{R}^n$ with stationary distribution π_f, and let P_u denote the distribution after one step from u. For every measurable set $S \subseteq \mathbb{R}^n$, define the *ergodic flow from* S by

$$\Phi(S) = \int_S P_u(\mathbb{R}_n \setminus S)\, d\pi_f(u).$$

We can read this quantity as follows: we select a random point X from distribution π and make one step to get Y. What is the probability that $X \in S$ and $Y \notin S$? It is easy to check that

$$\Phi(\mathbb{R}^n \setminus S) = \Phi(S).$$

For $0 < s \leq 1/2$, we define (as in [16]) the *s-conductance* of the Markov chain by

$$\Phi_s = \inf_{s < \pi_f(A) \leq 1/2} \frac{\Phi(A)}{\pi_f(A) - s},$$

and invoke Corollary 1.6(b) from [16] as a lemma. Let σ^m denote the distribution of the current position in the walk after t steps, and define the *starting error* by

$$H_s = \sup\{|\sigma^0(A) - \pi_f(A)| : \ \pi_f(A) \leq s\}$$

Then:

Lemma 9.1 *Let $0 < s \leq 1/2$. Then for every measurable $S \subseteq \mathbb{R}^n$, and every $m \geq 0$,*

$$|\sigma^m(S) - \phi_f(S)| \leq H_s + \frac{H_s}{s}\left(1 - \frac{1}{2}\Phi_s^2\right)^m.$$

After these preliminaries, we have to treat the ball walk and the hit-and-run walk separately.

9.1 Ball walk

In this section, we prove Theorem 2.2. Set

$$r = \frac{a\varepsilon^2}{2^{10}\sqrt{n}}, \qquad R = 8\sqrt{\mathsf{Var}(f)}\ln(1/\varepsilon), \qquad \varepsilon_1 = r^{1/2}n^{1/4}.$$

Let

$$K_0 = \{x \;:\; |x| < R,\ f(x) > e^{-2n-8\ln(1/\varepsilon)}M_f\}.$$

Then

$$\pi_f(K_0) \geq 1 - \frac{\varepsilon}{n^8}. \tag{29}$$

By Lemma 6.3,

$$\int_{\mathbb{R}^n} \hat{f}(x)\,dx \geq 1 - 64\varepsilon_1. \tag{30}$$

To apply Lemma 9.1, we need a lower bound on the ε_2-conductance of the walk, where $\varepsilon_2 = 256\varepsilon_1$. This will follow from the next lemma.

Lemma 9.2 *Let $\mathbb{R}^n = S_1 \cup S_2$ be a partition into measurable sets with $\pi_f(S_1), \pi_f(S_2) > \varepsilon_2$. Then*

$$\Phi(S_1) \geq \frac{r}{2^{15}\sqrt{n}R}(\pi_f(S_1) - \varepsilon_2)(\pi_f(S_2) - \varepsilon_2) \tag{31}$$

Proof. For $i \in \{1, 2\}$, let

$$\begin{aligned} S_i' &= \{x \in S_i \cap K_0 :\ P_x(S_{3-i}) < \frac{1}{64}\delta(x)\}, \\ \text{and } S_3' &= K_0 \setminus S_1' \setminus S_2'. \end{aligned}$$

First, suppose that $\pi_f(S_1') \leq \pi_f(S_1)/2$. Then the left hand side of the desired inequality is at least

$$\int_{u \in (S_1 \cap K_0)\setminus S_1'} \frac{\hat{f}(u)}{64 f(u)} f(u)\,du = \frac{1}{64}\pi_{\hat{f}}((S_1 \cap K_0) \setminus S_1').$$

Using (30) and (29),

$$\begin{aligned} \pi_{\hat{f}}((S_1 \cap K_0) \setminus S_1') &\geq \pi_f(S_1 \cap K_0 \setminus S_1') - \frac{\varepsilon_2}{4} \\ &\geq \pi_f(S_1) - \pi_f(\mathbb{R}^n \setminus K_0) - \pi_f(S_1') - \frac{\varepsilon_2}{4} \\ &\geq \frac{1}{2}(\pi_f(S_1) - \varepsilon_2). \end{aligned}$$

Hence,

$$\int_{S_1} P_u(S_2)\,d\pi_f \geq \frac{1}{128}(\pi_f(S_1) - \varepsilon_2)$$

which implies the lemma.

So we can assume that $\pi_f(S_1') \geq \pi_f(S_1)/2$, and similarly $\pi_f(S_2') \geq \pi_f(S_2)/2$. We now claim that

$$d_K(S_1', S_2') \geq \frac{r}{64R\sqrt{n}}. \tag{32}$$

Let $u \in S_1'$ and $v \in S_2'$. Then

$$d(P_u, P_v) > 1 - \frac{\max\{\delta(u), \delta(v)\}}{32},$$

and so by Lemma 7.1, one of the following holds:

$$d(u, v) \geq \frac{r}{8\sqrt{n}}, \tag{33}$$

or

$$d_f(u, v) \geq \frac{1}{8}. \tag{34}$$

By Lemma 6.14(b), inequality (33) implies that

$$d_K(u, v) \geq \frac{r}{64R\sqrt{n}}.$$

By Lemma 6.14(c), inequality (34) implies that

$$d_K(u, v) \geq \frac{d_f(u, v)}{6n + 48\ln(2/\varepsilon))} \geq \frac{1}{48n + 400\ln(2/\varepsilon))} > \frac{r}{64R\sqrt{n}}.$$

This proves (32). Now using the Isoperimetry Theorem 2.5 for d_K and $\pi_{\hat{f}}$, we get

$$\begin{aligned}
\pi_{\hat{f}}(S_3') &\geq d_K(S_1', S_2')\pi_{\hat{f}}(S_1')\pi_{\hat{f}}(S_2') \\
&\geq \frac{r}{64R\sqrt{n}}(\pi_f(S_1') - \frac{\varepsilon_2}{2})(\pi_f(S_2') - \frac{\varepsilon_2}{2}) \\
&\geq \frac{r}{256R\sqrt{n}}(\pi_f(S_1) - \varepsilon_2)(\pi_f(S_2) - \varepsilon_2).
\end{aligned}$$

We can now complete the proof of the lemma:

$$\begin{aligned}
\int_{S_1} P_u(S_2)\, d\pi_f &= \frac{1}{2}\Big(\int_{S_1} P_u(S_2)\, d\pi_f + \int_{S_2} P_u(S_1)\, d\pi_f\Big) \\
&\geq \frac{1}{2}\int_{S_3'} \frac{\hat{f}(u)}{64f(u)} f(u)du \\
&= \frac{1}{128}\pi_{\hat{f}}(S_3') \\
&\geq \frac{r}{2^{15}\sqrt{n}R}(\pi_f(S_1) - \varepsilon_2)(\pi_f(S_2) - \varepsilon_2).
\end{aligned}$$

□

To complete the proof of Theorem 2.2, let $s = \varepsilon/(2H)$. Then Lemma (9.2) implies

$$\Phi_s \geq \frac{r}{2^{15}\sqrt{n}R},$$

and trivially

$$H_s \leq H \cdot s,$$

so Lemma 9.1 gives that

$$|\sigma^m(S) - \pi_f(S)| \leq Hs + H \exp\left(-\frac{mr^2}{2^{31}nR^2}\right).$$

Hence Theorem 2.2 follows.

9.2 Hit-and-run

We recall the values of the following parameters:

$$t_0 = 8\ln(2/\varepsilon), \quad R = t_0\sqrt{n} \quad \text{and} \quad r = \frac{\varepsilon^2}{2^{18}\sqrt{n}}.$$

Lemma 9.3 *Let $S_1 \cup S_2$ be a partition of $\mathbb{R}^n$ into measurable sets with $\pi_f(S_1), \pi_f(S_2) > \varepsilon$. Then,*

$$\int_{S_1} P_u(S_2)\, d\pi_f \geq \frac{r}{2^{25}\sqrt{n}R}(\pi_f(S_1) - \varepsilon)(\pi_f(S_2) - \varepsilon) \tag{35}$$

Proof. For $i \in \{1, 2\}$, let

$$\begin{aligned} S_i' &= \{x \in S_i :\ P_x(S_{3-i}) < \frac{1}{2^{13}}\delta(x)\}, \\ \text{and} \quad S_3' &= \mathbb{R}^n \setminus S_1' \setminus S_2'. \end{aligned}$$

First, suppose that $\pi_{\hat{f}}(S_1') \leq \pi_{\hat{f}}(S_1)/2$. Then the left hand side of (35) is at least

$$\frac{1}{2^{13}}\int_{u \in S_1 \setminus S_1'} \frac{\hat{f}(u)}{f(u)} f(u)\, du = \frac{1}{2^{13}}\pi_{\hat{f}}(S_1 \setminus S_1') \geq \frac{1}{2^{14}}\pi_{\hat{f}}(S_1).$$

Corollary 6.5 implies that

$$\pi_{\hat{f}}(S_1) \geq \pi_f(S_1) - \frac{\varepsilon}{4}.$$

Hence,

$$\int_{S_1} P_u(S_2)\, d\pi_f \geq \frac{1}{2^{14}}(\pi_f(S_1) - \frac{\varepsilon}{4})$$

which implies (35).

So we can assume that $\pi_f(S_1') \geq \pi_f(S_1)/2$, and similarly $\pi_f(S_2') \geq \pi_f(S_2)/2$. Let W be the subset of $\mathbb{R}^n$ with $\alpha(u) > 2^{30}nR/r\varepsilon$. Then by Lemma 6.10,

$$\pi_f(W) \leq \frac{\varepsilon r}{2^{26}nR}. \tag{36}$$

By Lemma 7.2, for any two points $u_1 \in S_1' \setminus W$, $u_2 \in S_2' \setminus W$, one of the following holds:

$$
\begin{aligned}
d_f(u,v) &\geq \frac{1}{128\ln(3+\alpha(u))} \geq \frac{1}{2^{12}\ln(nR/\varepsilon)} && (37)\\
d(u,v) &\geq \frac{r}{2^{10}\sqrt{n}} && (38)
\end{aligned}
$$

Define

$$
\begin{aligned}
S_i'' &= S_i' \cap K \setminus W \text{ for } i = 1,2\\
\text{and } S_3'' &= K \setminus S_1'' \setminus S_2''.
\end{aligned}
$$

Then we get a lower bound on $d_K(u,v)$ for any $u \in S_1'', v \in S_2''$.

$$
d_K(u,v) \geq \frac{r}{2^{11}\sqrt{n}R}. \tag{39}
$$

Indeed, if (37) holds, Lemma 6.14(c) implies that

$$
d_K(u,v) \geq \frac{1}{6n+6t_0} \cdot \frac{1}{2^{12}\ln(nR/\varepsilon)} > \frac{r}{2^{11}\sqrt{n}R}.
$$

If (38) holds, then Lemma 6.14(b) implies that

$$
d_K(u,v) \geq \frac{1}{2R} \cdot \frac{r}{2^{10}\sqrt{n}} = \frac{r}{2^{11}\sqrt{n}R}.
$$

Using (39), we can apply Theorem 2.5 to $\hat{f}$ restricted to K to get

$$
\pi_{\hat{f}}(S_3'') \geq \frac{r}{2^{11}\sqrt{n}R}\pi_{\hat{f}}(S_1'')\pi_{\hat{f}}(S_2'').
$$

For $i = 1,2$, using Lemma 6.13 and (36),

$$
\pi_{\hat{f}}(S_i'') \geq \pi_f(S_i) - \frac{\varepsilon}{2}.
$$

Therefore,

$$
\pi_{\hat{f}}(S_3'') \geq \frac{r}{2^{11}\sqrt{n}R}(\pi_f(S_1) - \frac{\varepsilon}{2})(\pi_f(S_2) - \frac{\varepsilon}{2}).
$$

Using this,

$$
\begin{aligned}
\int_{S_1} P_u(S_2)\, d\pi_f &\geq \frac{1}{2}\int_{S_3''} \frac{\hat{f}(u)}{2^{13}f(u)} f(u)du - \pi_f(W)\\
&\geq \frac{1}{2^{14}}\pi_{\hat{f}}(S_3'') - \pi_f(W)\\
&\geq \frac{r}{2^{25}\sqrt{n}R}(\pi_f(S_1) - \varepsilon)(\pi_f(S_2) - \varepsilon)
\end{aligned}
$$

and (35) is proved. $\square$

Invoking Lemma 9.1 with $s = \varepsilon$, it follows that for every $m \geq 0$, and every measurable set S,

$$|\sigma^m(S) - \pi_f(S)| \leq H\varepsilon + H \exp\left(-\frac{mr^2}{2^{51}nR^2}\right) < H\varepsilon + H\left(-\frac{m\varepsilon^4}{2^{99}n^2\ln(2/\varepsilon)^2}\right).$$

Hence Theorem 2.3 follows on replacing ε by $\varepsilon/(2H)$.

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References

[1] D. Applegate and R. Kannan (1990): Sampling and integration of near log-concave functions, *Proc. 23th ACM STOC*, 156–163.

[2] D. Bertsimas and S. Vempala: Solving convex programs by random walks, *Journal of the ACM*, **51**(4) (2004), 540–556.

[3] J. Bourgain: Random points in isotropic convex sets, in: *Convex geometric analysis* (Berkeley, CA, 1996), 53–58, Math. Sci. Res. Inst. Publ., 34, Cambridge Univ. Press, 1999.

[4] A. Dinghas: Über eine Klasse superadditiver Mengenfunktionale von Brunn–Minkowski–Lusternik-schem Typus, *Math. Zeitschr.* **68** (1957), 111–125.

[5] M. Dyer, A. Frieze and R. Kannan: A random polynomial time algorithm for estimating volumes of convex bodies, *Proc. 21st Annual ACM Symp. on the Theory of Computing* (1989), 375–381.

[6] M. Dyer, A. Frieze and L. Stougie: talk at the Math. Prog. Symp., Ann Arbor, Mich. (1994).

[7] A. Frieze, R. Kannan and N. Polson: Sampling from log-concave distributions, *Ann. Appl. Probab.* **4** (1994), 812–837; Correction: Sampling from log-concave distributions, ibid. **4** (1994), 1255.

[8] A. Frieze and R. Kannan: Log-Sobolev inequalities and sampling from log-concave distributions, To appear in the Annals of Applied Probability

[9] M. Jerrum and A. Sinclair: Approximating the permanent, *SIAM Journal on Computing*, **18** (1989), 1149–1178.

[10] A. Kalai and S. Vempala: Efficient algorithms for universal portfolios, *Journal of Machine Learning Research*, **3** (2002), 423–440.

[11] R. Kannan and L. Lovász: A logarithmic Cheeger inequality and mixing in random walks, in: *Proc. 31st Annual ACM STOC* (1999), 282-287.

[12] R. Kannan, L. Lovász and M. Simonovits, "Random walks and an $O^*(n^5)$ volume algorithm for convex bodies," *Random Structures and Algorithms* 11 (1997), 1-50.

[13] R. Kannan, L. Lovász and M. Simonovits: "Isoperimetric problems for convex bodies and a localization lemma," *J. Discr. Comput. Geom.* **13** (1995) 541–559.

[14] L. Lovász (1992): How to compute the volume? *Jber. d. Dt. Math.-Verein, Jubiläumstagung 1990*, B. G. Teubner, Stuttgart, 138–151.

[15] L. Lovász: Hit-and-run mixes fast, *Math. Prog.* 86 (1998), 443-461.

[16] L. Lovász and M. Simonovits: Random walks in a convex body and an improved volume algorithm, *Random Structures and Alg.* **4** (1993), 359–412.

[17] L. Lovász and S. Vempala: Hit-and-Run from a Corner, *Proc. of the 36th ACM Symp. on the Theory of Comp.* (STOC '04) (2004), 310–314.

[18] L. Leindler: On a certain converse of Hölder's Inequality II, *Acta Sci. Math. Szeged* 33 (1972), 217–223.

[19] A. Prékopa: Logarithmic concave measures and functions, *Acta Sci. Math. Szeged* 34 (1973), 335–343.

[20] A. Prékopa: On logarithmic concave measures with applications to stochasic programming, *Acta Sci. Math. Szeged* 32 (1973), 301–316.

[21] A. Prékopa: *Stochastic Programming*, Akadémiai Kiadó, Budapest and Kluwer, Dordrecht 1995.

[22] M. Rudelson: Random vectors in the isotropic position, *J. Funct. Anal.* **164** (1999), 60–72.

[23] R.L. Smith (1984), Efficient Monte-Carlo procedures for generating points uniformly distributed over bounded regions, *Operations Res.* **32**, 1296–1308.

[24] Z.B. Zabinsky, R.L. Smith, J.F. McDonald, H.E. Romeijn and D.E. Kaufman: Improving hit-and-run for global optimization, *J. Global Optim.* **3** (1993), 171–192.