

The Colin de Verdière Number and Nullspace Embeddings

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Loosely speaking, an embedding of a graph G in $\mathbb{R}^r$ consists of an injective mapping $\psi : V \rightarrow \mathbb{R}^r$, and a correspondence from each edge $ij \in E$ to a simple curve in $\mathbb{R}^r$ with endpoints $\psi(i)$ and $\psi(j)$. Here a curve can be taken to be the image of a continuous injective function $\phi : [0, 1] \rightarrow \mathbb{R}^r$. An embedding in S^2 is defined similarly. In a planar embedding images of edges cross only at images of vertices shared by the edges.

A graph is planar if it has a planar embedding, outerplanar if it can be planarly embedded so that all the vertices are on the outer face, and linklessly embeddable if it can be embedded in $\mathbb{R}^3$ so that any two disjoint circuits form unlinked curves in $\mathbb{R}^3$ (see Figure 1 for an illustration).

These notes assume familiarity with embeddings and planarity. More details on planar graphs can be found in standard combinatorial optimization and graph theory textbooks, e.g. [KV02, Die00, Bol98].

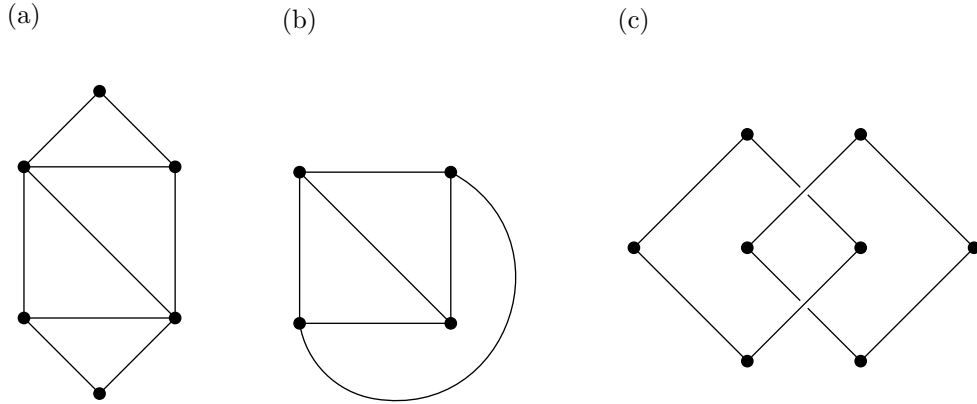


Figure 1: (a) an outerplanar graph; (b) K_4 , a graph which is not outerplanar; (c) two linked curves in $\mathbb{R}^3$.

1 Introduction

Consider an undirected graph $G = (V, E)$ with n vertices, and let $\mathcal{M}(G) \subseteq \mathbb{R}^{n \times n}$ be the set of symmetric matrices $M = (m_{ij})$ satisfying:

- (M1) For $i \neq j$, $m_{ij} < 0$ if $ij \in E$ and $m_{ij} = 0$ otherwise. (Note that there is no constraint on m_{ii} .)
- (M2) M has exactly one negative eigenvalue.

(M3) For any symmetric matrix $X = (x_{ij}) \in \mathbb{R}^{n \times n}$ with $x_{ij} = 0$ if $i = j$ or $ij \in E$, we have $MX = 0 \Rightarrow X = 0$. (This condition is also called the Strong Arnold Property.)

Let $\mathcal{M}'(G)$ be the set of matrices satisfying (M1) and (M2) only. Y. Colin de Verdière [CdV90, CdV93] introduced the parameter $\mu(G)$ denoting the maximum corank of any matrix in $\mathcal{M}(G)$:

$$\mu(G) := \max_{M \in \mathcal{M}(G)} \text{corank} M, \quad \mu'(G) := \max_{M \in \mathcal{M}'(G)} \text{corank} M. \quad (1)$$

Recall that the corank of a matrix is the dimension of its nullspace, $\text{corank} M := \dim \ker M$. A matrix $M \in \mathcal{M}(G)$ realizing the maximum corank is called a Colin de Verdière matrix for G . We first make a few remarks about the definition.

1. By the Perron-Frobenius theorem, any irreducible, nonnegative matrix has a unique largest eigenvalue. In the case of connected graphs, this implies that the negative eigenvalue in (M2) has multiplicity one.
2. Let A be the adjacency matrix of G and let $\lambda_1 > \lambda_2 > \dots > \lambda_r$ denote the eigenvalues of A . Then for $\lambda_1 > \lambda > \lambda_2$, consider the matrix $M = \lambda I - A$. M satisfies condition (M1) by the definition of the adjacency matrix. (M2) is satisfied since M has eigenvalues $\lambda - \lambda_1, \lambda - \lambda_2, \dots, \lambda - \lambda_r$, of which only the first is negative. Note that since M is nonsingular, $MX = 0 \Rightarrow X = 0$, implying condition (M3). In particular, for any graph G , $\mathcal{M}(G)$ is nonempty.

Furthermore, if we allow arbitrary entries along the diagonal of A so that the eigenvalues of A are distinct, then $M = \lambda_2 I - A$ will satisfy (M1) and (M2) for the same reasons as above (note that there are no restrictions on the diagonal entries of M in the definition). Also, since the corank of M is one, any matrix X satisfying $MX = 0$ has rank one. Since in addition, all the diagonal entries of X are zero, X must be the zero matrix, implying M satisfies condition (M3). In particular, $\mu(G) \geq 1$ for any $G \neq K_1$.

Examples:

1. Consider $\overline{K_n}$, the graph on n independent vertices. All entries of M except the diagonal entries must be zero. At least one diagonal entry must be negative, by condition (M2). Suppose we try to minimize the rank by putting more than one zero in the diagonal, say in positions m_{11} and m_{22} . Then the matrix $x_{12} = x_{21} = 1$, and $x_{ij} = 0$ for $\{i, j\} \neq \{1, 2\}$ violates condition (M3). Therefore, we can put only one zero in the diagonal, implying $\mu(\overline{K_n}) \leq 1$. By the previous remarks, we also have $\mu(\overline{K_n}) \geq 1$, so $\mu(\overline{K_n}) = 1$.
2. For the complete graph K_n , the rank of M is at least one by condition (M1), so $\mu(G) \leq n - 1$. Now, consider the matrix $M = -J_n$, i.e., the $n \times n$ matrix of all negative ones. Then M satisfies (M1) and (M2). Also, the only matrix X satisfying the condition in (M3) is the all zero matrix. Therefore, $M = -J_n$ is the Colin de Verdière matrix for K_n and $\mu(K_n) = n - 1$.

3. Let G be a path on vertices $1, 2, \dots, n$ in this order. For any matrix M satisfying condition (M1), delete the first column and last row of M . The resulting matrix is lower triangular with negative entries along the diagonal. Since the determinant is the product of the diagonal entries and is nonzero, this matrix is nonsingular. So the rank of M is at least $n - 1$, implying the corank is at most one, and therefore $\mu(G) = 1$.

A graph H is called a minor of G if H can be obtained from G by deletions and contractions of edges. Colin de Verdiere showed that $\mu(G)$ is minor-monotone.

Lemma 1. *If H is a minor of G , then $\mu(H) \leq \mu(G)$.*

Next, recall the following theorem of Robertson and Seymour [RS04].

Theorem 2. *Any minor-monotone property P of graphs can be characterized by a finite sequence of forbidden minors.*

It immediately follows that the set of graphs with $\mu(G) \leq \mu_0$ can be characterized by a finite sequence of forbidden minors. On the other hand, several planarity properties can also be characterized by forbidden minors:

Class of graphs	Forbidden minors
Paths	$K_3, K_{1,3}$
Outerplanar	$K_4, K_{2,3}$
Planar	$K_5, K_{3,3}$
Linklessly Embeddable	Petersen family

The following equivalences relate planarity and $\mu(G)$:

1. $\mu(G) \leq 1$ if and only if G is a disjoint union of paths;
2. $\mu(G) \leq 2$ if and only if G is outerplanar;
3. $\mu(G) \leq 3$ if and only if G is planar;
4. $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

Equivalences (1–3) are due to Colin de Verdiere [CdV90, CdV93], and (4) is due to Lovasz and Schrijver [LS98]. Later in this lecture (Section 2) we will show (3).

Now consider a basis $a_1, \dots, a_r$ of $\ker M$, let B be a matrix that has the vectors a_i as its columns, and let $u_1, \dots, u_n$ be its rows. We scale u_i as follows:

$$v_i := \frac{u_i}{\|u_i\|}, \quad w_i := \frac{u_i}{\pi_i}, \quad (2)$$

where $\pi = (\pi_1, \dots, \pi_n)$ is the eigenvector corresponding to the negative eigenvalue. (It will be shown later that π_i and $\|u_i\|$ are nonzero for matrices we consider.)

It turns out that v_i and w_i yield embeddings as follows [LS99]:

1. Let G be a path, and $M \in \mathcal{M}'(G)$ with $\text{corank} M = 1$. Then the mapping $i \rightarrow w_i$ together with the segments connecting w_i and w_j for $ij \in E$ yields an embedding of G on a line.

2. Let G be 2-connected outerplanar, and $M \in \mathcal{M}'(G)$ with $\text{corank} M = 2$. Then the mapping $i \rightarrow v_i$ together with the segments connecting v_i and v_j for $ij \in E$ yields an outerplanar embedding of G .
3. Let G be 3-connected planar, and $M \in \mathcal{M}'(G)$ with $\text{corank} M = 3$. Then the mapping $i \rightarrow v_i$ together with the geodesic curves on S^2 connecting v_i and v_j for $ij \in E$ yields a planar embedding of G on the sphere S^2 .

In Section 3 we will show (3). We note that if G when 3-connected, then $\mu(G) = \mu'(G) \leq 3$ [vdHLS95], in other words the Strong Arnold Hypothesis can be waived.

2 Proof that $\mu(G) \leq 3 \Leftrightarrow G$ is planar

Let $\text{supp}(x) := \{i \in V : x_i \neq 0\}$, and $\text{supp}^+(x) := \{i \in V : x_i > 0\}$, $\text{supp}^-(x) := \{i \in V : x_i < 0\}$. Before proceeding with the proof of the main theorem, we need the following three lemmas.

Lemma 3. *Let G be connected, and $x \in \ker M$. Then $\text{supp}^+(x)$ and $\text{supp}^-(x)$ are both nonempty.*

Proof. Since G is connected, Perron-Frobenius theory implies that the eigenvector π corresponding to the negative eigenvalue λ can be taken positive. But $\pi x = \pi Mx = 0$, and either $\text{supp}^+(x)$ or $\text{supp}^-(x)$ has to be nonempty, both are nonempty. $\square$

Lemma 4. *Let $x \in \ker M$. Then each vertex $i \in V \setminus \text{supp}(x)$ with a neighbor in $\text{supp}^+(x)$ has a neighbor in $\text{supp}^-(x)$, and vice-versa.*

Proof. Consider the i -th row of M :

$$0 = \sum_{j=1}^n m_{ij}x_j = \sum_{j \in \text{supp}^+(x)} m_{ij}x_j + \sum_{j \in \text{supp}^-(x)} m_{ij}x_j. \quad (3)$$

Now because $\sum_{j \in \text{supp}^+(x)} m_{ij}x_j < 0$, it follows that there is at least one $j \in \text{supp}^-(x)$ with $m_{ij} < 0$. $\square$

A vector $x \in \ker M$ has minimal support if there no other vector $y \in \ker M$ with $\text{supp}(y) \subsetneq \text{supp}(x)$. The following lemma is due to van der Holst [vdH95]:

Lemma 5. *Let G be connected, and $x \in \ker M$. Then $\text{supp}^+(x)$ and $\text{supp}^-(x)$ both span connected subgraphs of G .*

Proof. W.l.o.g. suppose that $\text{supp}^+(x)$ is disconnected, let I and J be two of its components, and let $K := \text{supp}^-(x)$. We will denote by $M_{I \times J}$ the submatrix of M spanned by the rows indices in I and the column indices in J .

Note that $m_{ij} = 0$ for $i \in J, j \in J$, and thus

$$M_{I \times I}x_I + M_{I \times K}x_K = 0, \quad (4)$$

$$M_{J \times J}x_J + M_{J \times K}x_K = 0. \quad (5)$$

Recall that π , the eigenvector corresponding to the negative eigenvalue λ can be taken positive, and let

$$\lambda := \frac{z_I^T x_I}{z_J^T x_J}, \quad y := \begin{cases} x_i, & i \in I, \\ -\lambda x_i, & i \in J, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Note that $z^T y = z_I^T x_I - \lambda z_J^T x_J = 0$, and

$$\begin{aligned} y^T M y &= y_I^T M_{I \times I} y_I + y_J^T M_{J \times J} y_J = x_I^T M_{I \times I} x_I + \lambda^2 x_J^T M_{J \times J} x_J = \\ &\quad - x_I^T M_{I \times K} x_K - \lambda^2 x_J^T M_{J \times K} x_K \leq 0. \end{aligned} \quad (7)$$

In the above, we again used the fact that $m_{ij} = 0$ for $i \in I, j \in J$, as well as $M_{I \times K}, M_{J \times K} \leq 0$, and $x_I > 0, x_J > 0, x_K < 0$.

Since M is symmetric and has exactly one negative eigenvalue, $\pi^T y = 0$ and $y^T M y \leq 0$ imply $M y = 0$, and $\text{supp}(y) \subsetneq \text{supp}(x)$, which is a contradiction. $\square$

Observation 6. *There is an intuitive geometric interpretation of van der Holst's lemma.*

Let B be a matrix having as columns a set of vectors forming a basis of $\ker M$, as defined in Section 1. Since $x \in \ker M$, we can write

$$x = Ba, \quad a \in \mathbb{R}^r, \quad (8)$$

and note that a lives in the same space as the vectors $u_i, i \in V$. Consider the hyperplane $ay = 0$, and note that $au_i = 0$ for all $i \in V \setminus \text{supp}(x)$, $au_i > 0$ for $i \in \text{supp}^+(x)$ and $au_i < 0$ for $i \in \text{supp}^-(x)$. Moreover, note that x has minimal support if and only if the hyperplane $ay = 0$ cannot be rotated to contain an additional vector u_i , in other words, $ay = 0$ is spanned by some vectors u_i .

Hence, van der Holst's lemma can be restated to say that any hyperplane that goes through the origin and is spanned by some vectors u_i divides the graph into two connected components. $\square$

We are now ready to prove Colin de Verdière's characterization of planar graphs; the second part of this proof is due to van der Holst [vdH95].

Theorem 7. *Let G be a 3-connected graph. Then G is planar if and only if $\mu(G) \leq 3$.*

Proof. First, suppose G is not planar. Then it must contain either K_5 or $K_{3,3}$. It is known that $\mu(K_5) = \mu(K_{3,3}) = 4$, and since $\mu(G)$ is minor-monotone, $\mu(G) \geq 4$.

Conversely, suppose G is planar, but $\mu(G) > 3$. Since adding edges does not decrease $\mu(G)$, we may assume that G is maximally planar. Since it is also 3-connected, there is a face ijk that is a triangle. We can take a matrix $M \in \mathcal{M}(G)$ with $\text{corank} M = 4$, as well as a vector $x \in \ker M$ with $x \neq 0, x_i = x_j = x_k = 0$, and with minimal support.

Since G is 3-connected, there are 3 vertex-disjoint paths from some vertices $u, v, w \in V \setminus \text{supp}(x)$ to i, j , and k respectively. Moreover, u, v , and w have neighbors in both $\text{supp}^+(x)$ and $\text{supp}^-(x)$. By contracting $\text{supp}^+(x), \text{supp}^-(x)$ and P_1, P_2, P_3 to a vertex each, we obtain a $K_{2,3}$ with the three vertices i, j, k of degree 2 on one face. This is a minor of G , but it cannot be planar since adding a new vertex inside the face ijk would yield a planar embedding of $K_{3,3}$. (For an illustration, see Figure 2.) $\square$

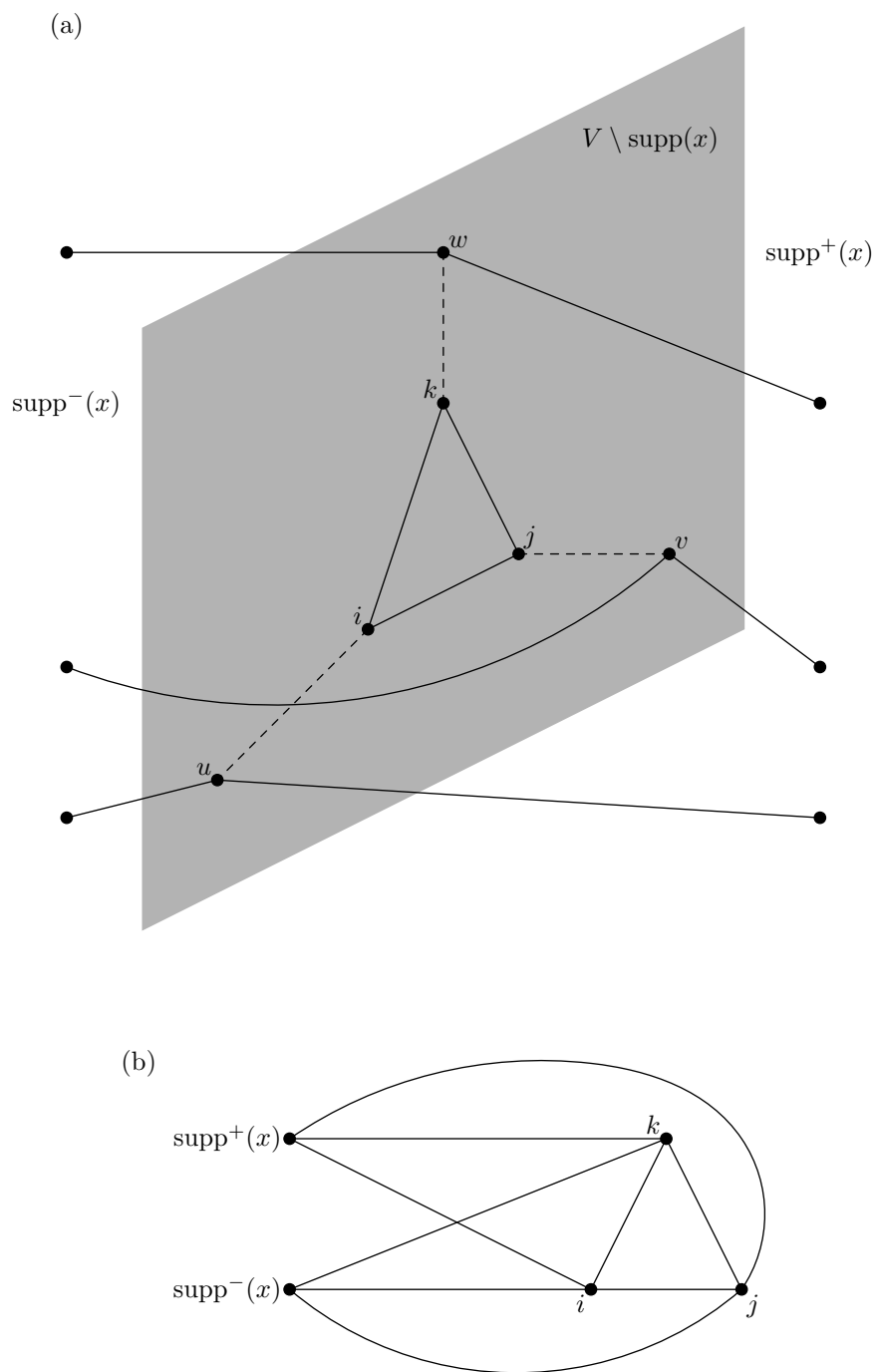


Figure 2: (a) the construction on G ; (b) the $K_{2,3}$ resulting from contraction.

3 Proof that the nullspace yields a planar embedding

In the proof, we will consider the following embeddings and mappings:

- $f : G \rightarrow S^2$, defining some fixed embedding of G onto S^2 . Since G is planar, we know that such an embedding exists, and we may assume w.l.o.g. that all edges are represented by unique geodesics on the sphere.
- $\psi : G \rightarrow S^2$, mapping $i \rightarrow v_i$ and edges $ij \in E$ to the geodesics connecting $\psi(i)$ and $\psi(j)$. We will show that $\psi|_V$ is injective, and the geodesics are unique.
- $\phi : S^2 \rightarrow S^2$, which we will construct partially, will map the embedding given by f to that given by ψ . By showing that ϕ is one-to-one, we will obtain that ψ is a planar embedding.

Before proceeding, we give an outline of the proof:

1. We show that ϕ can be defined to satisfy certain local properties w.r.t. vertices and faces.
2. We show that ϕ is continuous and locally one-to-one.
3. We prove that ϕ is (globally) k -to-one, and use this to show that it is (globally) one-to-one.

We begin with the local properties.

Lemma 8. *If i, j, k are vertices on a face of G , then u_i, u_j , and u_k are linearly independent.*

Proof. Similar to the proof of Theorem 7; see [LS99]. □

Lemma 9. *Denote by $1, \dots, k$ the vertices of a face F of G , in this cyclic order. Then $u_1, \dots, u_k$ are the extreme rays of the convex cone they generate in $\mathbb{R}^3$, also in this cyclic order.*

Proof. Consider two consecutive vertices on F , e.g. 1 and 2, and let H be the plane spanned by u_1, u_2 and 0. It suffices to prove that $u_3, \dots, u_k$ are all on the same side of H .

Take a vector $x \neq 0, x \in \ker M$ with $x_1 = x_2 = 0$. Lemma 8 implies that $3, \dots, k \in \text{supp}(x)$, hence $u_3, \dots, u_k$ are not on H . Suppose they are not on the same side of H . Consider vertex disjoint paths P_1 and P_2 starting in $V \setminus \text{supp}(x)$ and ending in 1 and 2 respectively. Contract $P_1, P_2, \text{supp}^+(x)$, and $\text{supp}^-(x)$ to a vertex each and note that since $u_3, \dots, u_k$ are not all on the same side of H , there is an edge from $\text{supp}^+(x)$ to $\text{supp}^-(x)$. Hence we obtain a K_4 planarly embedded with all vertices on one face, which is not possible. (For an illustration, see Figure 3.) □

Lemma 10. *Let $a \in V$ and denote by $1, \dots, k$ the vertices adjacent to a , in this cyclic order. Then the geodesics S^2 connecting v_a to $v_1, \dots, v_k$ issue from v_a in this cyclic order. Moreover, if we let $x \neq 0, x \in \ker M$ with $x_a = x_1 = 0$, then there are no h, i, j with $2 \leq h < i < j \leq k$ such that $x_h > 0, x_i \leq 0$, and $x_j > 0$.*

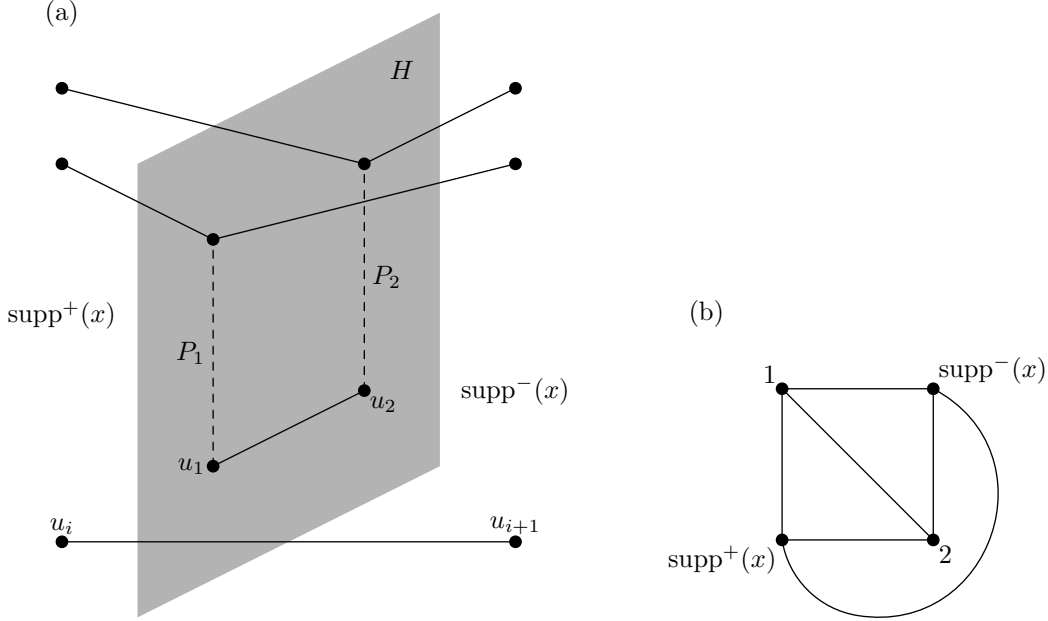


Figure 3: (a) the construction on G ; (b) the K_4 resulting from contraction.

Proof. See [LS99]. □

We are now ready to complete the proof.

Theorem 11. *Let G be a 3-connected planar graph, and let $M \in \mathcal{M}'(G)$ with $\text{corank } M = 3$. Then $\psi : G \rightarrow S^2$ yields a planar embedding of G on S^2 .*

Proof. Well-definiteness. Lemma 8 immediately implies that v_i are well-defined, since $u_i \neq 0$. Since two points on the sphere are connected by a unique geodesic unless they are at opposite ends of the sphere (in which case they are linearly dependent), Lemma 8 also implies that the edge images are well defined.

Definition of ϕ . We define ϕ so that it maps vertices to vertices, and edges to edges. Lemma 9 implies that we can extend ϕ to map faces to faces, and do this so that ϕ restricted to any vertex, edge, or face of $f(G)$ is continuous and one-to-one. One way to achieve this for faces is to triangulate each face, which can be done in the same way on both $f(G)$ and $\psi(G)$ since the cyclic order is preserved, and then map triangles to triangles.

Continuity of ϕ . Its definition implies ϕ is continuous on S^2 . To see this, take a sequence $\{x_i\}$ converging to some point $x \in S^2$, and note that it can be split into a finite number of sequences $\{x_i^k\}$, each converging to x , and each contained in a vertex, the interior of an edge, or the interior of a face. As a result, the images $\phi(x_i^k) \rightarrow \phi(x)$, hence $\phi(x_i) \rightarrow \phi(x)$.

Locally one-to-one, k -to-one. For points inside images of faces $f(F)$, this follows from the definition of ϕ . For images of vertices $f(a)$, note that there is a neighborhood N_F for each face $F, a \in F$ s.t. $\phi|_{N_F \cap \text{int } f(F)}$ is one-to-one. Similarly, there is a neighborhood N_e for each edge $e, a \in e$ s.t. $\phi|_{N_e \cap \text{int } f(e)}$ is locally one-to-one. Note that $f(a), \text{int } f(e)$ for

$a \in e, \text{int} f(F)$ for $a \in F$ are disjoint, and their union contains some neighborhood $N(f(a))$. On the other hand, $\psi(a), \text{int} \psi(e)$ for $a \in e, \text{int} \psi(F)$ for $a \in F$ are also disjoint, since for any face $F, a \in F$, all points of $\psi(F)$ are between the two geodesics corresponding to the two edges of the face emanating from a , due to Lemma 10. Hence

$$\phi \text{ restricted to } N(f(a)) \cap \left(\bigcap_{F: a \in F} N_F \right) \cap \left(\bigcap_{e: a \in e} N_e \right) \quad (9)$$

is one-to-one. The proof for edges is a simpler version of the proof for vertices, therefore ϕ is locally one-to-one. By Lemma 12, ϕ is k -to-one.

One-to-one. Consider an arbitrary 3-connected graph H planarly embedded on S^2 , with n vertices, m edges, and p faces. Preimages of edges do not cross, otherwise the locally one-to-one property implies that H is not planarly embedded. $\phi^{-1}(H)$ is connected, since otherwise one of the preimages of H would have to be strictly in the interior of a face of another preimage. This would imply that H is embedded so that some face is strictly contained in itself, which is a contradiction. Hence that $\phi^{-1}(H)$ is the embedding of a graph with kn vertices, km edges, and kp faces.

Using Euler's relation, from $\phi^{-1}(H)$, we obtain $kn - km + kp = 2$, and from H we obtain $n - m + p = 2$, hence $k = 1$. $\square$

Lemma 12. *Let $g : A \rightarrow B$ be a continuous locally one-to-one mapping. If its image B is connected, and g has a finite number of preimages for any $b \in B$, then g is k -to-one.*

Proof. Let $k(b) := |g^{-1}(b)|$ be the number of preimages of a point in B . It suffices to show that for any $b \in B$ and any sequence $\{b_i \in B\}$ converging to b , $\exists \lim_{i \rightarrow +\infty} k(b_i) = k(b)$. Assume $1 \leq k(b_i) \leq K$ (the case when $k(b_i)$ is unbounded is similar). If $k(b_i) \not\rightarrow k(b)$, then we can extract a convergent subsequence b'_i with $\lim_{i \rightarrow +\infty} k(b'_i) = k'(b) \neq k(b)$. Let $a_j, 1 \leq j \leq k$ be the k preimages of b , and let $N(a_j)$ be k neighborhoods of these preimages such that $g|_{N(a_j)}$ is one-to-one.

If $k'(b) > k(b)$, then, with the exception of a finite number of elements, we can divide $g^{-1}(b'_i)$ into $k'(b)$ distinct sequences. Each of them has to converge to some a_j , however, since $g|_{N(a_j)}$ are all one-to-one, at most $k(b)$ of them can do so. Therefore $k'(b) < k(b)$. Consider $N(b) := g(\cap_j N(a_j))$. Again since $g|_{N(a_j)}$ are one-to-one, each point in $N(b)$ has at least $k(b)$ preimages. However, all but a finite number of elements of $\{b'_i\}$ are in $N(b)$, hence $k'(b) = k(b)$.

This shows that $k(b) := |g^{-1}(b)|$ is continuous on B . Since $k(b) \in \mathbb{Z}$ and the image B is connected, $\exists k = k(b)$ for any $b \in B$. $\square$

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