CS 4540, Fall 2014

Homework 3

due: Wednesday, September 24, 2014 (at the start of class).

1. [Mitzenmacher-Upfal] Exercise 5.7.
   (For part (c): First write an expression for the probability that two bins receive the same number of balls, call that quantity P.)

2. [Mitzenmacher-Upfal] Exercise 5.11.

3. [Mitzenmacher-Upfal] Exercise 5.24.

4. Here we are going to prove the inequality we used in Monday's class when analyzing randomized QuickSort.

   (a) Prove that:
   $$k! \geq \left(\frac{k}{e}\right)^k.$$
   Use that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

   (b) Prove that:
   $$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

   (c) Prove that:
   $$2\binom{n}{k} \leq \binom{n}{k+1} \text{for } k \leq n/4.$$

   (d) Prove that:
   $$\sum_{i=0}^{k} \binom{n}{i} \leq 2\binom{n}{k} \text{for } k \leq n/4.$$

   (e) Let $X_1, \ldots, X_k$ be $0-1$ random variables representing k unbiased coin flips. Hence,
   $$\Pr\left[X_i = 1\right] = \Pr\left[X_i = 0\right] = 1/2.$$
   Prove that:
   $$\Pr\left[X_1 + X_2 + \cdots + X_k \leq k/4\right] \leq 2(.68)^{k/4}$$