CS 4540 Fall 2014, Homework 1
Out: Wednesday August 20, 2014
Due: Wednesday August 27, 2014

Homework Problems

1. Explain how to implement Karger's algorithm so that one run takes $O(n^2)$ time. Be sure to explain in detail how you implement the steps of the algorithm.

2. In the notation of class and Eric's handwritten notes, the event $\mathcal{E}_i$ is the event that the i-th contracted edge is not in the minimum cut. We showed that:

$$\Pr\left(\, \mathcal{E}_2 \,|\, \mathcal{E}_1 \,\right) \geq \left(1 - \frac{2}{n-1}\right).$$

   This says that the probability that the second edge contraction is not in the minimum cut, given that the first edge contraction is not in the minimum cut, is $\geq 1 - 2/(n-1)$. Did we use the conditioning? In other words, let $\overline{\mathcal{E}_1}$ denote the event that the first contracted edge is in the minimum cut, so it's the complement of the event $\mathcal{E}_1$. Is it true that:

$$\Pr\left(\, \mathcal{E}_2 \,|\, \overline{\mathcal{E}_1} \,\right) \geq \left(1 - \frac{2}{n-1}\right).$$

   This says that the probability that the second edge contraction is not in the minimum cut, given that the first edge contraction is in the minimum cut, is $\geq 1 - 2/(n-1)$. Explain why or why not this is true.

3. Number of min st-cuts: In Monday's class we will show that every graph has $\leq n^2$ cuts of minimum size. This will follow easily from our analysis of Karger's algorithm. This fact is not the case for minimum st-cuts. Define a family of n-vertex graphs with specified vertices s and t, and then prove for this family there are an exponential number of st-cuts of minimum size.

4. *Modified algorithm:* Suppose at each step we choose a random pair of vertices v and w (ignoring whether or not v and w are neighbors or not), and then we contract v and w. So instead of choosing a random edge to contract as in Karger's algorithm, we choose a random pair of vertices to contract. If v and w are not neighbors the contraction is defined the same as before we just don't have a self-loop to delete. Show that there are graphs for which this modified algorithm has exponentially small probability of finding a minimum cut.

   It suffices to show that the probability the algorithm succeeds is of the form $\leq p^n$ for some constant p where $1 > p > 0$. For example, it's OK to prove that it succeeds with probability $\leq (7/8)^{n/4}$. (Don't be fixated on having your analysis exactly matching these numbers $n/4, 7/8$, they are just examples that may not be correct for your example.)