NP-completeness:

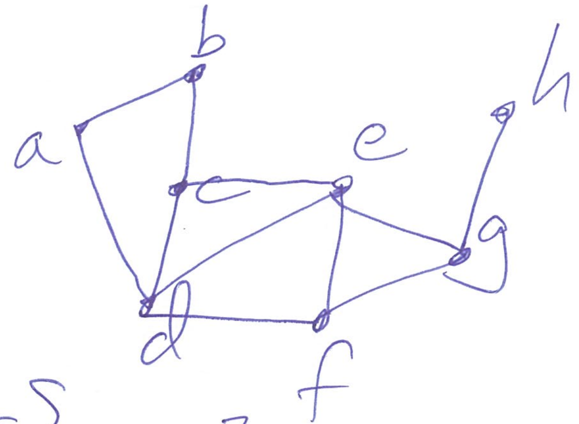
Last class: assuming SAT is NP-complete, we showed that 3SAT is NP-complete.

Today: Independent Set is NP-complete

Undirected $G=(V,E)$,

subset $S \subset V$ is an independent set if no edges are contained in S (in other words, for all $x,y \in S$, $(x,y) \notin E$)

Example:



$S=\{a,c,g\}$ is an independent set

$S'=\{a,c,f,h\}$ is an independent set of max size.

Independent Set problem:

input: undirected $G=(V,E)$ & goal g.

output: independent set S of size $|S| \geq g$
if one exists

NO otherwise.

Theorem: Independent Set problem is NP-complete.

Proof:

a) First we need to show that:

Independent Set $\in NP$

Given $G, g, \& S$, in $O(n^2)$ time we can check all pairs $x,y \in S$ to verify that (x,y) is not an edge.

b) We'll show $3SAT \rightarrow$ Independent Set.

Consider 3SAT input f with

n variables $x_1, \ldots, x_n$

& m clauses $C_1, \ldots, C_m$

Each clause has size ≤ 3.

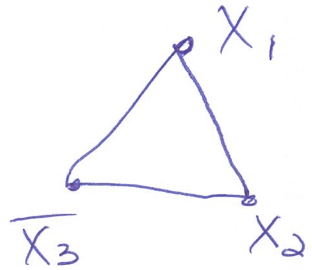
We'll create a graph G based on f.
f has m clauses so we'll set $g = m$.

Vertices of G correspond to literals of f.

For a clause $C = (a_1 \vee a_2 \vee a_3)$ for literals a_1, a_2, a_3
create 3 vertices corresponding to a_1, a_2, a_3
& add edges between all 3. (so a triangle)
This way independent set includes ≤ 1 of them.

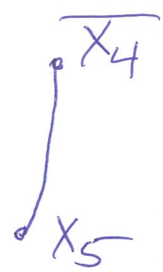
Example:

$C = (x_1 \vee \overline{x_3} \vee x_2) \Longrightarrow$



For a clause $C = (a_1 \vee a_2)$
do the same for 2 vertices

Example: $C = (\overline{x_4} \vee x_5) \Longrightarrow$



For a clause of size 1 $C = (a_1)$
create 1 new vertex corresponding
to literal a_1.

For each variable x_i,
add edges between all vertices
corresponding to x_i with all $\overline{x_i}$

These edges ensure that an independent set
does not include x_i & $\overline{x_i}$

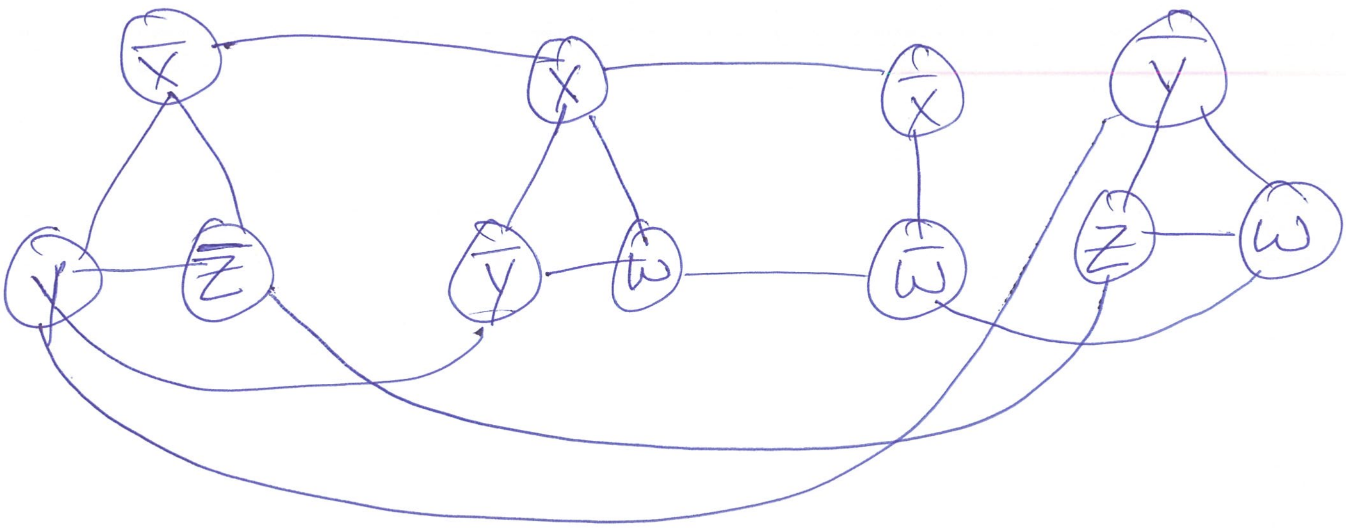
Thus an independent set corresponds to
a valid assignment

The edges within a clause ensure that
an indpt set includes ≤ 1 vertex per
clause. Thus to get size $= g = m$
we need exactly one vertex per clause.

Therefore an independent set of size g
corresponds to a satisfying assignment.

Example: Variables: x, y, w, z

$$f = (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee w) \wedge (\bar{x} \vee \bar{w}) \wedge (\bar{y} \vee z \vee w)$$



independent set

$S = \{ \bar{x}$ from 1st clause, $\bar{y}$ from 2nd clause, $\bar{w}$ from 3rd clause, $\bar{y}$ from last clause$\}$

Corresponds to assignment:

$x = F,\ y = F,\ w = F,\ z = ?$

which satisfies f.

Claim: f has a satisfying assignment $\iff$ G has an independent set of size $\geq g$

Proof:

($\Rightarrow$) Given satisfying assignment for f.

For each clause C, ≥ 1 literal satisfied.

Choose 1 satisfied literal & include its corresponding vertex in S.

$|S| = m = g$.

S includes 1 vertex per clause & does not include opposing literals

So S is an independent set.

($\Leftarrow$) Take an independent set S of size $\geq g$.

It has ≥ 1 vertex per clause, set that literal to T.

$\Rightarrow$ at least 1 satisfied literal per clause.

No contradictory literals (x_i & $\overline{x_i}$) since edges between them

Therefore we have a valid assignment that satisfies f.

∎

Clique: fully connected subgraph.

For $G=(V,E)$,

$S \subset V$ is a clique,

if for all $x,y \in S$, $(x,y) \in E$.

Want to find the largest clique.

Clique problem:

input: $G=(V,E)$ & goal g.

output: $S \subset V$ where S is a clique & $|S| \geq g$ if one exists

NO otherwise

Theorem: Clique is NP-complete.

Proof:

a) Clique $\in$ NP.

Given G, g & S in $O(n^2)$ time can check for all $x, y \in S$ that (x,y) is an edge.

& in $O(n)$ time can check that $|S| \geq g$.

Independent Set $\rightarrow$ Clique.

Key idea: Clique is opposite of independent set

Clique: all edges within S. Independent set: no edges within S

For $G=(V,E)$, let $\overline{G}=(V,\overline{E})$ where

$$\overline{E}=\{(x,y):(x,y)\notin E\}$$

So $(x,y)\notin E \Longleftrightarrow (x,y)\in\overline{E}$

Claim: S is an independent set in G $\Longleftrightarrow$ S is a clique in $\overline{G}$.

To show Independent Set $\rightarrow$ Clique.

Given $G=(V,E)$ & g for the independent set problem.

Let $\overline{G}$ & g be the input to the clique problem.

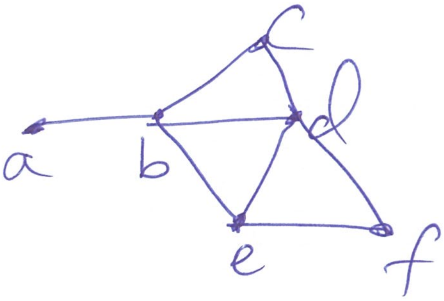
If we get a solution S for Clique then return the same S as the solution to the original independent set problem.

If we get NO then return NO for the independent set problem.

Vertex cover:

For $G=(V,E)$,
$S \subset V$ is a vertex cover if it
"covers every edge" meaning:
for every $(x,y) \in E$,
either $x \in S$ &/or $y \in S$.

Example:



$S = \{a, b, d, f\}$ is a vertex cover
Want to find the smallest VC.

Vertex Cover Problem:

input: $G=(V,E)$ & goal b.
output: Vertex cover S of size $|S| \leq b$
if one exists
NO otherwise

Theorem: Vertex Cover is NP-complete

Proof:

a) Vertex Cover $\in$ NP:

Given G, b & S in $O(n+m)$ time can check that every edge of G is covered by S & that $|S| \leq b$.

b) Independent Set $\rightarrow$ Vertex Cover.

In earlier example,

$S = \{a, b, d, f\}$ is a vertex cover

& $\overline{S} = \{c, e\}$ is an independent set.

This is true in general.

Claim: S is a vertex cover $\iff$ $\overline{S}$ is an independent set.

Proof:

($\Rightarrow$) Take VC S.

For each $(x,y) \in E$, x &/or y in S

Hence, ≤ 1 of x or y is in $\overline{S}$.

So edge contained in $\overline{S}$

& $\overline{S}$ is an independent set.

($\Leftarrow$) Take IS S.

For edge $(x,y) \in E$,

≤ 1 of x & y are in S,

So x &/or y is in $\overline{S}$

Hence $\overline{S}$ covers every edge (x,y).

$\blacksquare$

Independent Set $\longrightarrow$ Vertex Cover

For input G, g for IS

Use input $G, b = |V| - g$ for VC.

G has an indpt set of size $\geq g$ $\iff$ G has a vertex cover of size $\leq b$

If we get solution S for VC
then we return $\bar{S} = V - S$ as
the solution to the original IS problem.
If we get NO solution for VC,
then we return NO for IS.