Given a graph $G=(V,E)$ of maximum degree Δ & integer $k>0$, generate a random k-coloring of G.

Goal: Poly-time algorithm when $k>\Delta$.

Markov chain: Glauber dynamics = single-vertex update

let Ω = collection of proper vertex k-colorings of G.
Set X_0 = some proper vertex k-coloring.
From $X_t \in \Omega$,

1. Choose a vertex v uniformly at random (uar) from V & a color c uar from $\{1,\dots,k\}$
2. For all $w \neq v$ set $X_{t+1}(w) = X_t(w)$.
3. If no neighbors of v have color c then $X_{t+1}(v) = c$ else $X_{t+1}(v) = X_t(v)$.

Claim: When $k \geq \Delta+2$ the MC is ergodic.

Since P is symmetric then $\pi = \text{uniform}(\Omega)$ when $k \geq \Delta+2$.

Analyze mixing time = # of steps, from worst X_0, to get "close" to π.

Measure distance using total variation distance:

for a pair of distributions μ & ν on Ω,

$$d_{TV}(\mu,\nu) = \frac{1}{2}\sum_{\sigma\in\Omega} |\mu(\sigma)-\nu(\sigma)|$$
$$= \sum_{\sigma:\mu(\sigma)\geq\nu(\sigma)} \mu(\sigma)-\nu(\sigma)$$
$$= \max_{S\subset\Omega} \mu(S)-\nu(S).$$

for $\epsilon>0$, $X_0\in\Omega$,

$$T_{mix}^{X_0}(\epsilon) = \min\{t : d_{TV}(\underbrace{P^t(X_0,\cdot)}_{\text{distribution of } X_t \text{ given } X_0}, \pi)\leq\epsilon\}$$

& $$T_{mix}(\epsilon) = \max_{X_0\in\Omega} T_{mix}^{X_0}(\epsilon).$$

Let $T_{mix} = T_{mix}(1/4)$

then $T_{mix}(\epsilon) \leq T_{mix} \times \log\left(\frac{1}{\epsilon}\right)$. (Prove using coupling).

Coupling: a method to bound distance b/w distributions.

For μ & ν on Ω,

let ω be a distribution on $\Omega \times \Omega$.

Then ω is a coupling of μ & ν if:

for all $\sigma \in \Omega$, $\displaystyle\sum_{\tau \in \Omega} \omega(\sigma, \tau) = \mu(\sigma)$

& for all $\tau \in \Omega$, $\displaystyle\sum_{\sigma \in \Omega} \omega(\sigma, \tau) = \nu(\tau)$

(in words, for ω, the marginal in the 1st coordinate is μ
& in the 2nd coordinate is ν)

Choose $(X, Y) \sim \omega$

Then, $d_{TV}(\mu, \nu) \le \Pr(X \neq Y)$

Proof: $\Pr(X = Y) = \displaystyle\sum_{\sigma \in \Omega} \omega(\sigma, \sigma) \le \sum_{\sigma} \min\{\mu(\sigma), \nu(\sigma)\}$

& hence, $\Pr(X \neq Y) \ge 1 - \displaystyle\sum_{\sigma} \min\{\mu(\sigma), \nu(\sigma)\}$

$$= \sum_{\sigma} \mu(\sigma) - \min\{\mu(\sigma), \nu(\sigma)\}$$

$$= \sum_{\sigma : \mu(\sigma) \ge \nu(\sigma)} \mu(\sigma) - \nu(\sigma) = d_{TV}(\mu, \nu). \quad \blacksquare$$

Note, $\exists$ a coupling ω^* of μ & ν s.t.

$$d_{TV}(\mu,\nu) = Pr(X \neq Y).$$

Exercise: Prove this fact by construction.

Now consider a MC defined by P on Ω.
Make 2 copies (X_t) & (Y_t), with arbitrary X_0, Y_0.
From (X_t, Y_t), define (X_{t+1}, Y_{t+1}) so that:

$X_t \to X_{t+1}$ is distributed according to P

& $Y_t \to Y_{t+1}$ "

but they can be correlated.

Then, $d_{TV}(P^t(X_0,\cdot), P^t(Y_0,\cdot)) \leq Pr(X_t \neq Y_t)$

& if for all X_0, Y_0, $Pr(X_t \neq Y_t) \leq 1/4$

then by setting $Y_0 \sim \pi$ we have:

$$T_{mix} \leq t.$$

Toy example: Random walk on the n-dimensional hypercube.

$$\Omega = \{0,1\}^n = n\text{-bit vectors.}$$

From $X_t \in \{0,1\}^n$:

1. Choose i uar from $\{1,2,\dots,n\}$ & b uar from $\{0,1\}$.
2. for all $j \neq i$, set $X_{t+1}(j) = X_t(j)$.
3. Set $X_{t+1}(i) = b$.

Lemma: $T_{mix} = O(n \log n)$

Proof: For a pair $X_t, Y_t \in \{0,1\}^n$ define a coupling $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ by: Using the same random i & b.

Let $H_t = |\{i : X_t(i) \neq Y_t(i)\}|$

$\cancel{E[H_{t+1}]}$

$$E[H_{t+1} \mid X_t, Y_t] \leq H_t - \frac{H_t}{n} = H_t\left(1 - \frac{1}{n}\right)$$

$$\Pr(X_t \neq Y_t) \leq E[H_t] \leq H_0\left(1-\frac{1}{n}\right)^t \leq ne^{-t/n} \leq \tfrac{1}{4} \text{ for } t = n\ln(4n)$$

Back to colorings:

For k-colorings $X_t, Y_t \in \Omega$,

Choose same v & c for attempted update

Let $H_t = |\{w \in V : X_t(w) \neq Y_t(w)\}|$

$= \#$ of vertices that X_t & Y_t differ on.

Let $D_t = \{w \in V : X_t(w) \neq Y_t(w)\}$

& thus $H_t = |D_t|$

& $A_t = V \setminus D_t = \{w \in V : X_t(w) = Y_t(w)\}$

For $w \in V$, Let $a_t(w) = |A_t \cap N(w)|$ & $d_t(w) = |D_t \cap N(w)|$

$$\Pr(v \in A_{t+1} \mid v \in D_t) \geq \frac{k - 2\Delta + a_t(v)}{nk}$$

$$\Pr(v \in D_{t+1} \mid v \in A_t) \leq \frac{2d_t(v)}{nk}$$

$$E[H_{t+1} \mid X_t, Y_t] \leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} - \sum_{v \in D_t} \frac{k - 2\Delta + a_t(v)}{nk}$$

$$(\text{for } k \geq 3\Delta + 1) \leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} + \sum_{v \in D_t} \frac{-1 - \Delta - a_t(v)}{nk}$$

$$\leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} + \sum_{v \in D_t} \frac{-1 - 2a_t(v)}{nk}$$

So we have:

$$E[H_{t+1} \mid X_t, Y_t] \le H_t + \frac{1}{nk}\left[\sum_{v \in A_t} 2d_t(v) + \sum_{v \in D_t} -1 - 2a_t(v)\right]$$

Note, $\displaystyle\sum_{v \in A_t} d_t(v) = \sum_{v \in D_t} a_t(v)$

hence,

$$E[H_{t+1} \mid X_t, Y_t] \le H_t - \frac{|D_t|}{nk} = H_t\left(1 - \frac{1}{nk}\right)$$

Thus, ~~E[~~

$$\Pr(X_t \neq Y_t) \le E[H_t] \le H_0\left(1 - \frac{1}{nk}\right)^t$$

$$\le n e^{-t/nk}$$

$$\le \frac{1}{4} \text{ for } t = nk\log(4n)$$

when $k \ge 3\Delta + 1$.

This proves $T_{mix} = O(nk\log n)$ when $k > 3\Delta$.

How to improve?

Couplings compose:

Consider distributions μ, ν, ω on Ω,

& coupling α of μ & ν

coupling β of ν & ω

then

$\gamma = \alpha \circ \beta$ is a coupling of μ & ω.

Choose σ from μ

then apply α to choose τ from ν,

then apply β to choose η from ω.

Note,

$$\gamma(\sigma, \eta) = \sum_{\tau} \frac{\alpha(\sigma, \tau)\beta(\tau, \eta)}{\nu(\tau)}$$

Define a coupling for all pairs of colorings X_t, Y_t

where $H(X_t, Y_t) = H_t = 1$,

so they only differ on 1 vertex.

For arbitrary, ~~X_t, Z~~ W_t, Z_t

Define a sequence of colorings $X_t^0, X_t^1, \ldots, X_t^l$

where $X_t^0 = W_t$, $X_t^l = Z_t$

& $H(X_t^i, X_t^{i+1}) = 1 \quad \forall i$

Note, $l = H(W_t, Z_t)$

Define a coupling for all pairs X_t, Y_t

where $H(X_t, Y_t) = 1$

and the coupling satisfies:

$$E[H(X_{t+1}, Y_{t+1})] \leq 1 - \frac{1}{nk}.$$

Then, for arbitrary W_t, Z_t,

Consider the path $X_t^0, \ldots, X_t^\ell$

by composing couplings we have a coupling $(W_t, Z_t) \to (W_{t+1}, Z_{t+1})$

&

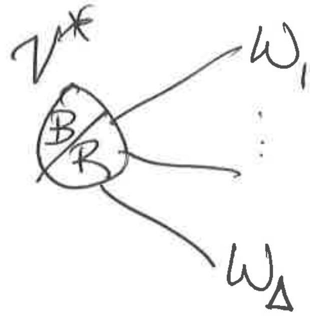
$$E[H(W_{t+1}, Z_{t+1})] \le E\left[\sum_{i=0}^{\ell-1} H(X_{t+1}^i, X_{t+1}^{i+1})\right]$$

$$\le \sum_i E[H(X_{t+1}^i, X_{t+1}^{i+1})]$$

$$\le \sum_i \left(1 - \frac{1}{nk}\right)$$

$$= H_t(1 - 1/nk).$$

So it suffices to analyze pairs that differ on 1 vertex.



~~Couple B in $X_t(v)$~~
~~with R in $Y_t(v)$~~
~~& vice-versa~~

for $w \in N(v^*)$,

Couple B in X_t with R in Y_t
R in X_t with B in Y_t

everything else, (v,c) is same for X_t, Y_t.

Note,

$$E[H(X_{t+1}, Y_{t+1})] = 1 - \frac{(k-\Delta)}{nk} + \frac{\Delta}{nk}$$

$$= 1 - \frac{(k-2\Delta)}{nk} \leq 1 - \frac{1}{nk}$$

for $k > 2\Delta$. ∎