CS 6550: Randomized Algorithms **Spring 2019**

Lecture 21: 2-SAT, MC Basics, and Page Rank

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21.1 2-SAT

We can solve a 2-SAT problem in a polynomial time by reducing it to finding strongly connected components of a directed graph. Alternatively, we can solve it via a randomized algorithm. Here is a simple algorithm that solves 2-SAT.

21.1.1 A randomized algorithm

This randomized algorithm finds a satisfying assignment of a 2-SAT problem and outputs it.

Algorithm 1: Algorithm

input : 2-SAT problem with clauses $C_1, \ldots, C_n$
output: a satisfying assignment σ, if exists

start with an arbitrary assignment σ_0;
for $i \leftarrow 1$ **to** $k \cdot n^2$ **do**
  take an unsatisfied clause C of σ_{i-1};
  choose a random literal in C and satisfy it;
  call the new assignment σ_i;
  if σ_i *satisfies the problem* **then**
    output σ_i

21.1.2 Analysis

Fix a satisfying assignment, and call it τ. Let σ_t be the assignment of the algorithm at the t-th iteration. Let $X_t = \#$ of variables that agree between σ_t and τ. If $X_t = n$, then the algorithm found a satisfying assignment. Also, X_t is a random walk in $\{0, \ldots, n\}$.

Claim 21.1 $Pr(X_{t+1} = i + 1 | X_t = i) \geq \frac{1}{2}$

Proof: Consider the unsatisfied clause C that is updated in $X_t \rightarrow X_{t+1}$. We know that τ satisfies C. Hence, one or more of the 2 variables in C have opposite assignments between τ and σ_t. Then, one or less of the 2 variables agree between σ_t and τ. ■

Consider an unbiased walk y_t such that $Pr(y_{t+1} = i+1|y_t = i) = Pr(y_{t+1} = i-1|y_t = i) = \frac{1}{2}$. Couple with X_t so that

- If $y_t = n$, then $X_t = n$ or σ_t is a satisfying assignment.
- If $y_{t+1} = y_t + 1$, then $X_{t+1} = X_t + 1$.

Let's define

- T_j = time to reach $X_t = n$ starting from $X_0 = j$.
- $h_j = \mathbb{E}[T_j]$.

Claim 21.2 $h_0 \leq n^2$

Proof: We have the following recurrence relation

$$h_j = 1 + \frac{1}{2}h_{j+1} + \frac{1}{2}h_{j-1}$$

It follows that

$$\begin{aligned} h_j &= 1 + \frac{1}{2}h_{j+1} + \frac{1}{2}h_{j-1} \\ \iff 2h_j &= 2 + h_{j+1} + h_{j-1} \\ \iff h_j - h_{j+1} &= h_{j-1} - h_j \end{aligned}$$

Given the base case $h_0 - h_1 = 1$, $h_j - h_{j+1} = 2j + 1$. Then,

$$\begin{aligned} h_0 &= (h_0 - h_1) + (h_1 - h_2) + \cdots + (h_{n-1} - h_n) \\ &= \sum_{i=0}^{n-1}(h_i - h_{i+1}) \\ &= \sum_{i=0}^{n-1}(2i+1) \\ &= 2(\sum_{i=0}^{n-1} i) + n \\ &= \frac{2n(n-1)}{2} + n \\ &= n^2 - n + n \\ &= n^2 \end{aligned}$$

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Consequentially,

$$h_0 = \begin{cases} O(n^2), & \text{biased} \\ O(n \log n), & \text{unbiased} \end{cases}$$

21.2 Markov Chain

21.2.1 Definitions

Definition 21.3 *$Pr(X_{t+1}|X_t = i_t, X_{t-1} = i_{t-1}, \cdots, X_0 = i_0) = Pr(X_{t+1} = j|X_t = i_t)$ is called the Markov property.*

Definition 21.4 *If a sequence of random variables $X_1, X_2, \ldots, X_n$ satisfies the Markov property, X_t is called a Markov chain on $\{0, 1, \ldots, n\}$.*

Definition 21.5 *Let X_t be a Markov chain on a state space $\Omega = \{1, 2, \ldots, N\}$. Then, the transition matrix $P \in \mathbb{R}^{N \times N}$ is defined by*

$$P(i, j) = Pr(X_{t+1} = j | X_t = i)$$

Also, it follows that

$$P^t(i, j) = Pr(X_t = j | X_0 = i)$$

Definition 21.6 *A Markov chain is called irreducible if $\forall i, j \in \Omega$, $\exists t$ such that $P^t(i, j) > 0$. In other words, the graph on P^t is one strongly connected component.*

Definition 21.7 *A Markov chain is called aperiodic if $\forall i \in \Omega$, its period = 1. The period of a state i is defined by $gcd(t : p^t(i, i) > 0)$.*

Definition 21.8 *A Markov chain is ergodic if and only if it is both irreducible and aperiodic.*

Definition 21.9 *A stationary distribution π of a Markov chain is a distribution of state probabilities satisfying $\pi P = \pi$.*

Example:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.2 & 0 & 0.5 & 0.3 \\ 0 & 0.3 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0.3 \end{bmatrix}$$

Then,

$$P^{20} = \begin{bmatrix} 0.244190 & 0.244187 & 0.406971 & 0.104652 \\ 0.244187 & 0.244186 & 0.406975 & 0.104651 \\ 0.244181 & 0.244185 & 0.406984 & 0.104650 \\ 0.244195 & 0.244188 & 0.406966 & 0.104652 \end{bmatrix} \text{ and } \pi = \lim_{t \to \infty} P^t \approx \begin{bmatrix} 0.2442 \\ 0.2442 \\ 0.4070 \\ 0.10465 \end{bmatrix}$$

21.2.2 Properties

Theorem 21.10 *An ergonic, finite Markov chain has a unique stationary distribution π. Also, for all $x_0 \in \Omega, j \in \Omega, \lim_{t \to \infty} Pr(X_t | X_0) = \pi(j)$.*

In order to find a stationary distribution π, we need Gaussian Elimination. However, $|\Omega|$ is usually very big.

Claim 21.11 *If P is symmetric, then $\pi = uniform(\Omega)$.*

Proof: Let's verify that for $\pi(i) = \frac{1}{N}, \pi P = \pi$.

$$\begin{aligned} (\pi P)(i) &= \sum_{k \in \Omega} \pi(k) P(k, i) \\ &= \frac{1}{N} \sum_{k \in \Omega} P(k, i) \\ &= \frac{1}{N} \sum_{k \in \Omega} P(i, k) && (P \text{ is symmetric}) \\ &= \frac{1}{N} && (P \text{ is stochastic}) \end{aligned}$$

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Claim 21.12 *P is reversible with respect to* π *if*

$$\forall i, j \in \Omega, \pi(i)P(i,j) = \pi(j)P(j,i).$$

In the equation above, π *is a stationary distribution.*

Proof:

$$\begin{aligned}(\pi P)(i) &= \sum_{k \in \Omega} \pi(k)P(k,i) \\ &= \sum_{k \in \Omega} \pi(i)P(i,k) \\ &= \pi(i) \sum_{k \in \Omega} \\ &= \pi(i)\end{aligned}$$

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Example: Consider a random walk on a d-regular undirected graph G. For $edge(i,j)$,

$$P(i,j) = P(j,i) = \frac{1}{d}.$$

So, it is symmetric and

$$\pi(i) = \frac{1}{n} \text{ for } n = |V|.$$

Now, consider a random walk on a non-regular undirected graph G. Then,

$$\pi(i) = \frac{d(i)}{z}$$

where $d(i)$ = degree of i, and $z = \sum_j d(j) = 2m$. It follows that

$$\pi(i)P(i,j) = \frac{d(i)}{z} \cdot \frac{1}{d(i)} = \frac{1}{z} = \pi(j)P(j,i).$$

21.3 PageRank

PageRank is a method to assign "importance" to webpages.

21.3.1 Problem Statement

Consider a graph G, where V = webpages and E = directed edges corresponding to hyperlinks.

- Idea 1: A link is a citation, so count the number of in-edges.
- Idea 2: Weight outgoing links by the number of hyperlinks on t. So, if page x has d outgoing links, then each gets $\frac{1}{d}$ of a citation. Hence, it is like a random walk

$$\pi(y) = \sum_{x : \vec{xy} \in E} \frac{1}{d(x)}$$

- Idea 3: Weight a page by its $\pi(x)$, hence:

$$\pi(y) = \sum_{x : \vec{xy} \in E} \frac{\pi(x)}{d(x)}$$

This corresponds to the stationary distribution of the random walk on the web graph. The stationary distribution π is not necessarily unique because the graph may not be ergodic. In order to make it ergodic,

1. Choose $0 < \alpha < 1$.
2. From page $x \in V$,
   - with prob α, choose a random out-edge.
   - with prob $1 - \alpha$, choose a random vertex in the whole graph.

Then, G is clearly ergodic and has a unique stationary distribution π. Also, π is the PageRank vector.

References

[1] Norris, James R. Markov Chains. In *Cambridge University Press*, 1997.