CS37501-1 Randomized Algorithms **Winter 2003**

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A hypergraph $H = (V, E)$ is said to have property B if there exists a 2-coloring of the vertices V such that no edge in E is monochromatic. The phrase property B was coined for Bernstein who originated the study of these combinatorial structures in 1908. In the following let $H = (V, E)$ be a d-uniform hypergraph and let $n = |V|$ and $m = |E|$. It is easy to see that if $m < 2^{d-1}$ then H has property B. Indeed, the expected number of monochromatic edges in a uniformly random 2-coloring is $m2^{-d+1} < 1$ and hence there is a coloring without a monochromatic edge. Erdös [2] constructed a d-uniform hypergraph with $O(2^d d^2)$ edges which does not have property B. Assume that n is even. Let $\chi : V \to \{\text{red}, \text{blue}\}$ be any coloring and e a random subset of V of size d. The probability that e is monochromatic in χ is at least

$$\frac{\binom{n/2}{d}}{\binom{n}{d}} \geq \frac{1}{2^d}\left(\frac{n-2d}{n-d}\right)^d \geq \frac{1}{2^d}\exp(-d^2/(n-2d)).$$

For m independent random subsets of V of size d the probability that none of them is monochromatic in χ is at most

$$\left(1 - \frac{1}{2^d}\exp(-d^2/(n-2d))\right)^m \leq \exp\left(-\frac{m}{2^d}\exp(-d^2/(n-2d))\right).$$

By the union bound the probability that there exists a coloring χ such that none of the random subsets is monochromatic in χ is at most

$$2^n \exp\left(-\frac{m}{2^d}\exp(-d^2/(n-2d))\right). \tag{5.1}$$

For $n = d^2 + d$ and $m > 2^d(d^2 + 2d)e\ln 2$ the value of (5.1) is smaller than 1 and hence there is a choice of m edges of size d such that the resulting hypergraph does not have property B.

Let $r(d)$ be the maximal m such that any d-uniform hypergraph with at most m edges has property B. From what we saw we know $2^{d-1} \leq r(d) \leq O(2^d \cdot d^2)$. Erdös and Lovász [3] conjectured that $r(d) = \Theta(2^d \cdot d)$. A lower bound of $r(d) = \Omega(d^{1/3}2^d)$ was shown by Beck in 1978 [1] (see also Alon and Spencer for a cleaner presentation of his proof). This was only recently improved by Radhakrishnan and Srinivasan [4]. We will show their improved lower bound $r(d) = \Omega(2^d \cdot \sqrt{d/\log d})$. They consider the following algorithm for finding a valid 2-coloring of a hypergraph $H = (V, E)$.

1. Let χ_0 be a random 2-coloring and let $v_1, \dots, v_n$ be a random permutation of the vertices in V. Let $Y_1, \dots, Y_n$ be i.i.d. 0, 1-random variables with $P(Y_i = 1) = p$ where p will be determined later.
2. For i from 1 to n do the following. If there is an edge $e \in E$ such that $v_i \in e$ and e is monochromatic in both χ_0 and χ_{i-1} and $Y_i = 1$ then let χ_i be the coloring which differs from χ_{i-1} in the color of v_i. Otherwise let $\chi_i = \chi_{i-1}$.

We will show that for a good choice of p the coloring χ_n produced by the algorithm is valid with non-zero probability for hypergraphs with not too many edges. To make the analysis simpler we will use the following process to generate a random permutation of V. To each $v \in V$ we assign an independent random variable X_v which is uniform in the interval $[0, 1]$ and then sort the elements in V in the increasing order of X_v.

Let $e \in E$. Consider the event that the edge e is all red in χ_n. There are two possibilities, either e is all red in χ_0 or it is not. The event that e is all red in χ_0 and in χ_n has probability at most

$$2^{-d}(1-p)^d \leq 2^{-d}\exp(-pd) \tag{5.2}$$

Now assume that e is not all red in χ_0 and it is all red in χ_n. Let $w \in e$ be the last vertex in e which was recolored and let f be the edge which was used by the algorithm to justify the recoloring of w. We will say that e blames f. Note that f was all blue in χ_0 and hence all vertices in $e \cap f$ get recolored. If a vertex of f is recolored then f is no longer monochromatic and hence cannot be used to justify a recoloring of a vertex. Hence $e \cap f = \{w\}$. Let S be the set of vertices of e which are blue in χ_0. Let us estimate the probability of the conditional event α_z (conditioned on $X_w = z$) that e is all red in χ_n, the set of vertices of e which are blue in χ_0 is S, and that e blames f. For α_z to happen the following independent events must occur:

- $f \cup S$ is all blue and $e \setminus S$ is all red in χ_0. Since $|e \cap f| = 1$, this event happens with probability 2^{-2d+1}.
- All of $S \setminus \{w\}$ changed colors before w: This requires that $X_v \leq X_w$ and $Y_v = 1$ for all $v \in S \setminus \{w\}$. This event happens with probability $(zp)^{|S|-1}$.
- None of $f \setminus \{w\}$ changed colors before w: This requires that $X_v \geq X_w$ or $Y_v = 0$ for all $v \in f \setminus \{w\}$. This event happens with probability $(1 - pz)^{d-1}$.
- Finally, for w to change colors, we need $Y_w = 1$, which has probability p.

Hence the probability of the event that e is all red in χ_n and it blames f is upper bounded by

$$\begin{aligned} p2^{-2d+1} \int_0^1 (1-pz)^{d-1} \sum_{\substack{S \subseteq e: \\ w \in S}} (pz)^{|S|-1} \mathrm{d}z &= p2^{-2d+1} \int_0^1 (1-pz)^{d-1}(1+pz)^{d-1} \mathrm{d}z \\ &= p2^{-2d+1} \int_0^1 (1-p^2z^2)^{d-1} \mathrm{d}z \\ &\leq p2^{-2d+1} \int_0^1 \mathrm{d}z \\ &= p2^{-2d+1}. \end{aligned}$$

Finally, the probability that some e blames some f is at most

$$2m^2 p 2^{-2d+1} \tag{5.3}$$

Combining (5.2) and (5.3) we obtain that the probability that there exists an edge in E which is monochromatic in χ_n is at most

$$2m2^{-d} \exp(-dp) + 4m^2 p 2^{-2d}.$$

Letting $m = k2^d$ (recall we are trying to maximize k) this simplifies to:

$$2k \exp(-pd) + 4k^2 p.$$

For $p = \frac{\ln d}{2d}$ we have

$$\frac{2k}{\sqrt{d}} + \frac{2k^2 \ln d}{d} \tag{5.4}$$

When $k \leq \frac{\sqrt{d}}{2\sqrt{\ln d}}$, the value of (5.4) is smaller than 1 (for sufficiently large d), and hence the probability that the algorithm succeeds is > 0. Hence H has property B.

References

[1] J. Beck. On 3-chromatic hypergraphs. *Discrete Math.*, 24(2):127–137, 1978.

[2] P. Erdős. On a combinatorial problem. II. *Acta Math. Acad. Sci. Hungar*, 15:445–447, 1964.

[3] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[4] J. Radhakrishnan and A. Srinivasan. Improved bounds and algorithms for hypergraph 2-coloring. *Random Structures Algorithms*, 16(1):4–32, 2000.