CS 7535 Markov Chain Monte Carlo Methods **Fall 2017**

Lecture 3: August 29

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3.1 Permanent in #P-Complete

- NP: Decision Problems - yes/no answers. $\exists witness$
- #P: Counting Problems - how many witnesses?

NP-problem: $X = \Sigma^* \times \Sigma^* \to \{0, 1\}$

1. $X(I, W)$ can be computed in poly-time
2. $\exists$ poly-time $p(x)$ where if $X(I, W) = I$ then $|W| \leq p(|I|)$

#P-problem: $f_x(I) = |\{W : X(I, W) = 1\}|$

Definition 3.1 ***Parsimonious*** - *preserves the number of solutions*

$A \subseteq B$ is parsimonious if $f_A(I) = f_B(g(I))$
note: I for A, $g(I)$ for B

We will cover the following reductions: #SAT is #P-complete $\leq$ #3SAT is #P-complete $\leq$ #Exact-3-Cover is #P-complete $\leq$ #Permanent is #P-complete

For some $n \times n$ matrix A:

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i \in [n]} A(i, \sigma(i)) \tag{3.1}$$

$$per(A) = \sum_{\sigma \in S_n} \prod_{i \in [n]} A(i, \sigma(i)) \tag{3.2}$$

For some 0-1 matrix A where edges are $edge(r_i, c_j) \leftrightarrow A(i, j) = 1$
i.e. no connections within the same row/column

If we have a 0-1 matrix then the $per(A) = \#$ of perfect matching in bipartite G_A

3.2 (0-1)-perm is #P-complete

(0-1)-perm = #Bi-Perf-Match
input: bipartite G
output: $|P|=$ # of perfect matchings

Perm = #Weight-Bi-Perf-Match
input: bipartite G with edge weights
output:

$$\sum_{p\ \in \wp} w(P) = \sum_{p \in \wp} \prod_{e \in P} w(e) \tag{3.3}$$

#Weighted-Bi-Match
input: bipartite G w/ weights
output:

$$\sum_{M\ \in \Omega} w(M) = \sum_{M \in \Omega} \prod_{e \in \Omega} w(e) \tag{3.4}$$

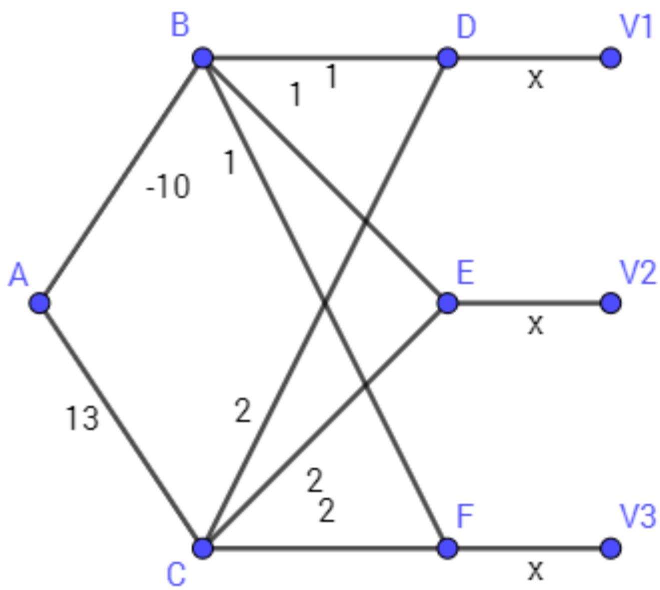
Note: $\Omega =$ all matchings in G

#Exact-3-Cover
input: set $X = \{x_1, ..., x_n\}$ & $Y \subseteq \binom{Y}{3}$
output: # of $Z \subseteq Y$ where $x \in X$ is covered exactly once.

Moving forward we will show #Exact-3-Cover $\leq$ #Weighted-Bi-Matching $\leq$ #Weight-Bi-Perf-Match $\leq$ #d-Bi-Perf-Match $\leq$ #(d-1)-Bi-Perf-Match.

note: d in the above context means $\leq$ d distinct edge weights $\neq 1$

We begin by constructing a gadget H



Ω = all matchings of H $w(\Omega) = \sum_{M \in \Omega} w(M) = 4(1 + x^3)$

take input (x, y) for #Exact-3-Cover for each $i \in X$, create verticies v_i & w_i and edge $e(v_i, w_i)$ with weight $= 1$

$\forall i, j, k \in Y$ add gadget H and identify V_1 with U_i, V_2 with U_j, and V_3 with U_k the total weight of matchings in $G = 4^M S$.

Now Ω = all matchings in G, $\Omega' \subset \Omega$ = matchings with cover all V_i's.

$w(\Omega \setminus \Omega') = 0$ Take $M \in \Omega \setminus \Omega'$ and suppose U_i is not covered in M then W_i is not covered $(U_i, W_i) \notin M$ and $M' = MU(U_i, V_i)$ also not that $W(M') = W(M)$.

The outcomes of this are:

1. U_i is not covered in M
2. W_i is covered in M

$\hat{M}$ where W_i is covered $\hat{M} = \hat{M} \setminus (U_i, W_i)$

$\hat{M} \in b, \hat{M}' \in a, M \in a, M \in b$

Take $M \in \Omega'$ and fix matchings on U_i's, ever U_i has one gadget covering it with x^3 edges

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Lecture 4: August 31

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4.3 (0,1)-Perm is #P-Complete

Last time, we claimed that (0,1)-perm is #P-Complete and showed:

1. Given a 0-1 matrix, $per(A)$ = #Bi-Perf-Match.

2. #Exact-3-Cover $\leq$ #Weighted-Bi-Matching

According to 3.2, we will finish our proof that (0-1)-perm is #P-Complete by showing:

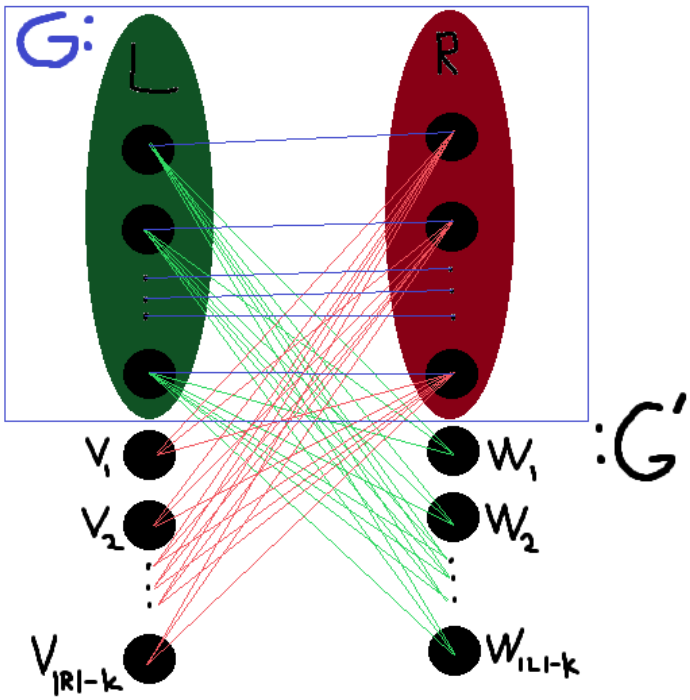
1. #Weighted-Bi-Matching $\leq$ #Weight-Bi-Perf-Match and

2. #d-Bi-Perf-Match $\leq$ #(d-1)-Bi-Perf-Match

Our first claim begins with a bipartite graph G = (L $\cup$ R, E), where L denotes G's "left half" and R denotes G's "right half".

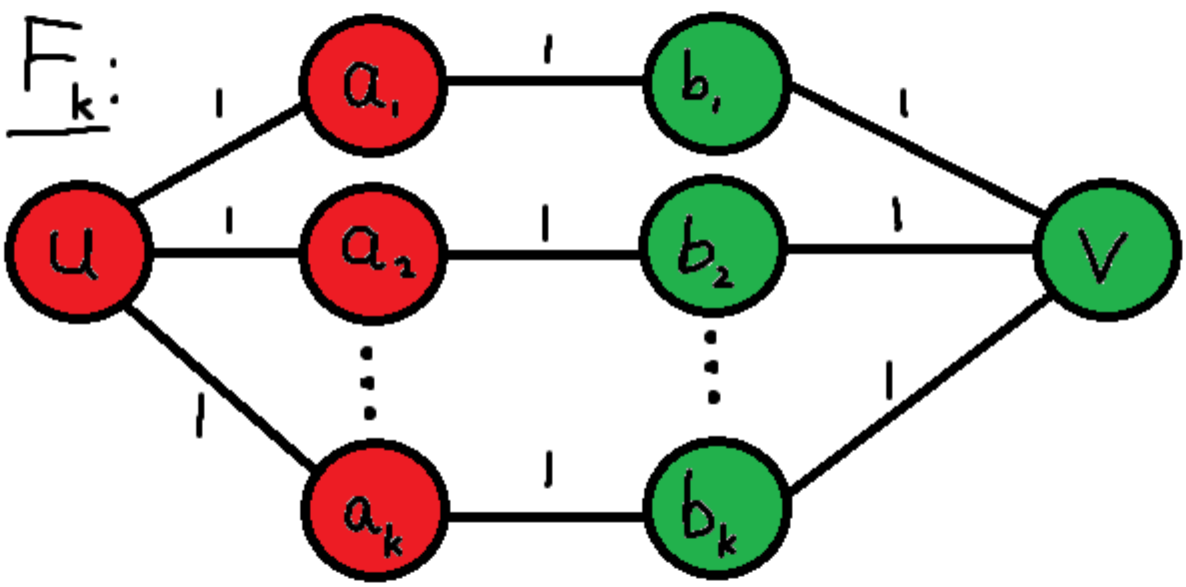
Let $|L|$ denote the number of vertices in L, $|R|$ denote the number of vertices in R, and k denote an integer such that: $0 \leq k \leq min\{|L|, |R|\}$.

Form a graph G' by adding vertices $v_1, v_2, ..., v_{|R|-k}$ connected to each vertex in R and $w_1, w_2, ..., w_{|L|-k}$ connected to each vertex in L.



Each matching of size k in G corresponds to $(|L|-k)!\ (|R|-k)!$ perfect matchings in G'.
Hence, #Weight-Bi-Perf-Match(G') = (#Weighted-Bi-Match(G)) $(|L|-k)!\ (|R|-k)!$.
This proves #Weighted-Bi-Matching $\leq$ #Weight-Bi-Perf-Match.

To prove our second claim, let A denote the corresponding adjacency matrix for a given graph G; we want Perm(A). Choose an edge weigh in G other than 0 or 1, call it α, to replace in A with a variable x. Perm(A) is now a degree n polynomial in x.
Now, choose an integer k, $0 \leq$ k $\leq$ n, and replace each α weighted edge in A with the F_k gadget pictured below for that k.



Now, Perm(A) is a polynomial in k, and the number of non-1 edge weights in G decreased by one.
By changing the size of the gadget, one can solve p(α) by evaluating p(k) at 0, 1, ..., n.
Fact: A degree n polynomial is uniquely determined at n + 1 points via Gaussian elimination of its matrix form (our adjacency matrix A).
This completes the reduction: #d-Bi-Perf-Match $\leq$ #(d-1)-Bi-Perf-Match. This also completes the proof that (0,1)-Perm is #P-Complete.

FPRAS = Fully Polynomial Randomized Approximation Scheme
Given a graph, some error tolerance ϵ, and the correct output P for an instance of (0,1)-Perm, an FPRAS produces OUT for (0,1)-Perm in time poly(n, $\frac{1}{\epsilon}$) such that:

$$Pr((1-\epsilon)|P| \leq OUT \leq (1+\epsilon)|P|) \geq \frac{3}{4} \tag{4.5}$$

Can we do better? Yes. Run FPRAS multiple times and take the median of the results. How many times do we need to run FPRAS to attain a success probability $\geq (1-\delta)$?
Let $X_i = 1$ if $Y_i \in (1-\epsilon)|P|$ or 0 otherwise. Assume the X_i's are iid.

$$\hat{X} = \sum_{i=1}^{k} X_i \tag{4.6}$$

And:

$$E[\hat{X}] \geq \frac{3k}{4} \tag{4.7}$$

Then:

$$Pr(OUT \notin (1 \pm \epsilon|P|) \leq Pr(X < \frac{k}{2}) \tag{4.8}$$

(if $\geq \frac{1}{2}$ of the trials are "good", then the median is "good", according to Professor Vigoda)

$$\mu = \frac{k\epsilon}{4} \tag{4.9}$$

For some ϵ:

$$\frac{3k}{4} = \frac{k\epsilon}{4} \tag{4.10}$$

Solving for ϵ:

$$\epsilon = \frac{1}{3} \tag{4.11}$$

Use Chernoff Bounds to solve for δ:

$$Pr(|x-\mu| > \epsilon\mu) \leq 2e^{\frac{-\mu\epsilon^2}{3}} \tag{4.12}$$

Setting μ = E[X] gives:

$$\begin{aligned} &Pr(OUT \notin (1 \pm \epsilon|P|) \\ &\leq Pr(|x-\mu| \geq \frac{k}{4}) \\ &\leq 2e^{-(\frac{1}{3}^2)(\frac{3}{4})k(\frac{1}{3})} \\ &\leq 2e^{(\frac{1}{9})(\frac{1}{4})k} \\ &\leq 2e^{\frac{k}{36}} \end{aligned} \tag{4.13}$$

So if we set:

$$k = 36ln(\frac{2}{\delta}) \tag{4.14}$$

We obtain:

$$\begin{aligned} &Pr(OUT \notin (1 \pm \epsilon)|P|) \\ &\leq 2e^{\frac{-36ln(\frac{2}{\delta})}{36}} \\ &\leq 2e^{-ln(\frac{2}{\delta})} \\ &\leq 2\frac{\delta}{2} = \delta \end{aligned} \tag{4.15}$$

So to run FPRAS successfully with error $\leq \delta$ takes O(log($\frac{1}{\delta}$)) iterations.

FPAUS: Fully Polynomial Almost Uniform Sampler
Generate from a graph and some $\delta > 0$ some perfect matching p $\in$ P from a distribution π.
Satisfies

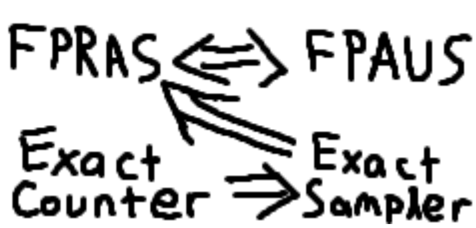
$$d_t v(u, \pi) \leq \delta \tag{4.16}$$

in time

$$poly(n, log(\frac{1}{\delta})) \tag{4.17}$$

where u = uniform(P) and:

$$d_t v = (\frac{1}{2})(\sum_{x \in P} |u(x) - \pi(x)| = max_{S \subseteq P}(u(s) - \pi(s))) \tag{4.18}$$



References