The diameter and mixing time of critical random graphs.

Yuval Peres

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Joint work with: Asaf Nachmias.

Background

The Erdos and Rényi random graph $G(n, p)$ is obtained from the complete graph on n vertices by retaining each edge with probability p and deleting it with probability $1 - p$, independently of all other edges. Let $\mathcal{C}_1$ denote the largest component of $G(n, p)$.

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Theorem (Erdos and Rényi, 1960)

If $p = \frac{c}{n}$ then

1. *If $c < 1$ then $|\mathcal{C}_1| = O(\log n)$ a.a.s.*
2. *If $c > 1$ then $|\mathcal{C}_1| = \Theta(n)$ a.a.s.*
3. *If $c = 1$, then $|\mathcal{C}_1| \sim n^{2/3}$ (proved later by Bollobas, and also Luczak)*

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2. *If $c > 1$ then diam$(\mathcal{C}_1) = \Theta(\log n)$ a.a.s.*

The mixing time of the lazy random walk on a graph G is

$$T_{\mathrm{mix}}(G) = T_{\mathrm{mix}}(G, 1/4) = \min\{t : \|\mathbf{p}^t(x,\cdot) - \pi(\cdot)\| \leq 1/4, \forall x \in V\},$$

where $\|\mu - \nu\| = \max_{A \subset V} |\mu(A) - \nu(A)|$ is the total variation distance.

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Theorem (Fountoulakis and Reed & Benjamini, Kozma and Wormald)

If $p = \frac{c}{n}$ where $c > 1$, then the random walk on $\mathcal{C}_1$, the largest component of $G(n,p)$ (the unique component of linear size), has

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Question: [Benjamini, Kozma and Wormald] What is the order of the mixing time of the random walk on the largest component of the critical random graph $G(n, \frac{1}{n})$?

Main Result

Theorem (Nachmias, P.)

Let $\mathcal{C}_1$ denote the largest connected component of $G(n, \frac{1}{n})$. Then for any $\epsilon > 0$ there exists $A = A(\epsilon) < \infty$ such that for all large n,

- $$\mathbf{P}\Big(\mathrm{diam}(\mathcal{C}_1) \notin [A^{-1}n^{1/3}, An^{1/3}]\Big) < \epsilon\,,$$
- $$\mathbf{P}\Big(T_{\mathrm{mix}}(\mathcal{C}_1) \notin [A^{-1}n, An]\Big) < \epsilon\,.$$

This answer the question of Benjamini, Kozma and Wormald.

Remark. This extends for p in the "critical window", i.e. $p = \frac{1+\lambda n^{-1/3}}{n}$.

A general theorem

Let G be a d-regular graph and $p \in (0,1)$ and consider the random subgraph G_p is obtained from bond percolation with parameter p. We write $\mathcal{C}$ for a connected component of G_p.

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If $p \leq \frac{1}{d-1}$ then for any $\epsilon > 0$ there exists $A = A(\epsilon) < \infty$ such that

1. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{E}(\mathcal{C})| > An^{2/3}\Big) < \epsilon\,,$
2. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > \beta n^{2/3}\,,\ \mathrm{diam}(\mathcal{C}) \notin [A^{-1}n^{1/3}, An^{1/3}]\Big) < \epsilon\,,$
3. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > \beta n^{2/3}\,,\ T_{\mathrm{mix}}(\mathcal{C}) \notin [A^{-1}n, An]\Big) < \epsilon\,.$

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Remark: Later we will see how to relax the assumption $p \le \frac{1}{d-1}$.

Applications of general theorem

The general Theorem implies the Theorem about $G(n, \frac{1}{n})$ because

$$\liminf_n \mathbf{P}(|\mathcal{C}_1| > \beta n^{2/3}) \to 1\,, \quad \text{as } \beta \to 0\,,$$

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By the general Theorem, the same estimates for the diameter and the mixing time hold for:

1. Random d-regular graphs on n vertices when $p \leq \frac{1}{d-1}$ (Nachmias, P., 2006).
2. Cartesian product of two complete graphs (van der Hofstad and Luczak, 2006 and Borgs, Chayes, van der Hofstad, Slade and Spencer, 2005).

The diameter of critical random graphs

Let Γ be an infinite d-regular tree with root ρ and let Γ_p be the outcome of p-bond percolation on Γ. Let $\mathcal{C}(\rho)$ be the component containing ρ in Γ_p. Define

$$\mathcal{L}_k = \Big\{ u \in \mathcal{C}(\rho) \,:\, d_\Gamma(\rho, u) = k \Big\} .$$

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Theorem (Kolmogorov, 1938)

If $p = \frac{1}{d-1}$ *then*

$$\mathbf{P}\Big(|\mathcal{L}_k| > 0 \Big) \leq \frac{c}{k}\,.$$

Some definitions

For a vertex $v \in G$ let $\mathcal{C}(v)$ be the component containing v in G_p. Let $d_p(u, v)$ denote the distance between u and v in G_p. Define

$$B_p(v, k) = \{u \in \mathcal{C}(v) : d_p(v, u) \le k\} ,$$

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As G is d-regular we can couple such that

$$|B_p(v,k)| \le \sum_{j=0}^{k} |\mathcal{L}_j|,$$
$$|\partial B_p(v,k)| \le |\mathcal{L}_k|.$$

Upper bound on the diameter

If a vertex $v \in V$ satisfies $\mathrm{diam}(\mathcal{C}(v)) > R$, then $|\partial B_p(v, \lceil R/2 \rceil)| > 0$, thus by our coupling

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Write

$$X = \Big|\{v \in V : |\mathcal{C}(v)| > M \text{ and } \mathrm{diam}(\mathcal{C}(v)) > R\}\Big|.$$

Then we have $\mathbf{E}X \leq \frac{2cn}{R}$. So we have

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$$\mathbf{P}\quad\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > M \text{ and } \mathrm{diam}(\mathcal{C}) > R\Big) \le \mathbf{P}(X > M) \le \frac{2cn}{MR},$$

and taking $M = \beta n^{2/3}$ and $R = An^{1/3}$ concludes the proof.

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Write

$$Y = \Big|\{v \in V : |\mathcal{C}(v)| > M \text{ and } \mathrm{diam}(\mathcal{C}(v)) < r\}\Big|\,.$$

Lower bound on the diameter (continued)

We learn that $\mathbf{E} Y \leq \frac{2rn}{M}$. As before this gives

$$\begin{aligned} & \mathbf{P} \quad \left(\exists \mathcal{C} \in \mathbf{CO}(G_p) \text{ with } |\mathcal{C}| > M \text{ and } \mathrm{diam}(\mathcal{C}) > r\right) \\ \leq & \quad \mathbf{P}(Y > M) \leq \frac{2rn}{M^2} . \end{aligned}$$

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and taking $M = \beta n^{2/3}$ and $r = A^{-1} n^{1/3}$ concludes the proof.

Upper bound on the size of components

For any $v \in V$ we have

$$\{|\mathcal{C}(v)| > M\} \subset \{|\mathcal{C}(v)| > M \text{ and } \mathrm{diam}(\mathcal{C}(v)) \leq r\} \cup \{\mathrm{diam}(\mathcal{C}(v)) > r\}\,.$$

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Write

$$Z = \Big|\{v \in V : |\mathcal{C}(v)| > M\}\Big|\,.$$

We have shown that

$$\mathbf{E}Z \le \left(\frac{2c}{r} + \frac{2r}{M}\right) n\,.$$

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Thus,

$$\mathbf{P}\Big(|\mathcal{C}_1| > M\Big) \le \mathbf{P}\Big(Z > M\Big) \le \left(\frac{2c}{rM} + \frac{2r}{M^2}\right) n\,,$$

and taking $M = An^{2/3}$ and $r = \sqrt{A}n^{1/3}$ concludes the proof.

Upper bound on the mixing time

The upper bound $T_{\mathrm{mix}}(\mathcal{C}_1) \leq O(n)$ follows from

Lemma

Let $G = (V, \mathcal{E})$ be a graph. Then the mixing time of a lazy simple random walk on G satisfies

$$T_{\mathrm{mix}}(G, 1/4) \leq 8|\mathcal{E}(G)|\mathrm{diam}(G)\,.$$

Lower bound on the mixing time

Let $\mathcal{R}(u \leftrightarrow v)$ denote the effective resistance between u and v.

Lemma (Tetali 1991)

For a lazy simple random walk on a finite graph where each edge has unit conductance, we have

$$\mathbf{E}_v \tau_z = \sum_{u \in V} \deg(u) [\mathcal{R}(v \leftrightarrow z) + \mathcal{R}(z \leftrightarrow u) - \mathcal{R}(u \leftrightarrow v)] .$$

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Lemma (Nash-Williams 1959)

If $\{\Pi_j\}_{j=1}^J$ are disjoint cut-sets separating v from z in a graph with unit conductance for each edge, then the effective resistance from v to z satisfies

$$\mathcal{R}(v \leftrightarrow z) \geq \sum_{j=1}^{J} \frac{1}{|\Pi_j|} \,.$$

Lower bound on the mixing time (continued)

For a graph $G = (V, \mathcal{E})$, write $d_G(x, y)$ for the graph distance between x and y. For any vertex v, let

$$B(v, r) = B_G(v, r) = \{u \in v : d_G(u, v) \le r\},$$

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- An edge e between $\partial B(v, j-1)$ and $\partial B(v, j)$ is called a **lane** for (v, r) if it there is a path with initial edge e from $\partial B(v, j-1)$ to $\partial B(v, r)$ that does not return to $\partial B(v, j-1)$.

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- Say that a level j (with $0 < j < r$) has L **lanes** for (v, r) if there are at least L edges between $\partial B(v, j-1)$ and $\partial B(v, j)$ which are lanes for (v, r).

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- Say that a level j (with $0 < j < r$) has L **lanes** for (v, r) if there are at least L edges between $\partial B(v, j-1)$ and $\partial B(v, j)$ which are lanes for (v, r).
- Let $k < r$. A vertex v is called L-**lane rich** for (k, r), if more than half of the levels $j \in [k/2, k]$ have L lanes for (v, r).

Lower bound on the mixing time (continued)

Lemma

Let $G = (V, \mathcal{E})$ be a graph and let $v \in V$. Suppose that $|B(v,h)| \geq m$, that v is not L-lane rich for (k,r), that $|\mathcal{E}(B(v,r))| < \frac{|\mathcal{E}(G)|}{3}$ and that $h < \frac{k}{4L}$. Then

$$T_{\mathrm{mix}}(G) \geq \frac{mk}{12L}\,.$$

Another general theorem

Recall the definitions:

$$B_p(v,k) = \{u \in \mathcal{C}(v) : d_p(v,u) \le k\}\,,$$

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Theorem (Nachmias, P.) If $p \in (0,1)$ satisfies

(i) $\mathbf{E}|\mathcal{E}(B_p(v,k))| \le c_1 k$,

(ii) $\mathbf{P}(|\partial B_p(v,k)| > 0) \le c_2/k$,

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then for large enough A

1. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{E}(\mathcal{C})| > An^{2/3}\Big) < \epsilon\,,$
2. $\mathbf{P}\Big(\exists \mathcal{C} \text{ with } |\mathcal{C}| > \beta n^{2/3}\,,\ \mathrm{diam}(\mathcal{C}) \notin [A^{-1}n^{1/3}, An^{1/3}]\Big) < \epsilon\,,$
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An open question

Consider $T_n^d = \{0, \ldots, n-1\}^d$, the d-dimensional discrete torus with side n and consider p-bond percolation on it. Let $V = n^d$ denote the volume of this graph.

Theorem (Borgs, Chayes, van der Hofstad, Slade and Spencer 2005)

Let d be large but fixed, and $n \to \infty$. Then there exists some p_c such that there exists a "critical window" around p_c. I.e., for all $p = p_c + \Theta(V^{-1/3})$ we have

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Question: Does this p_c have properties (i) and (ii)?