Topics:

- Optimization
- Convolutional Layers

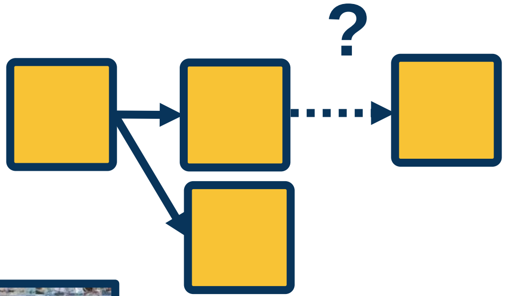
CS 4644-DL / 7643-A
ZSOLT KIRA

- **Assignment 2 out – Due Feb 17th (grace period until Feb 19th)**
  - Implement convolutional neural networks

- **Meta Lectures**: Data wrangling OH unfortunately not recorded ☹
  - Issue fixed, sorry!
  - Next one 02/21

There are still many design decisions that must be made:

- **Architecture**
- **Data Considerations**
- **Training and Optimization**
- **Machine Learning Considerations**



Optimizers

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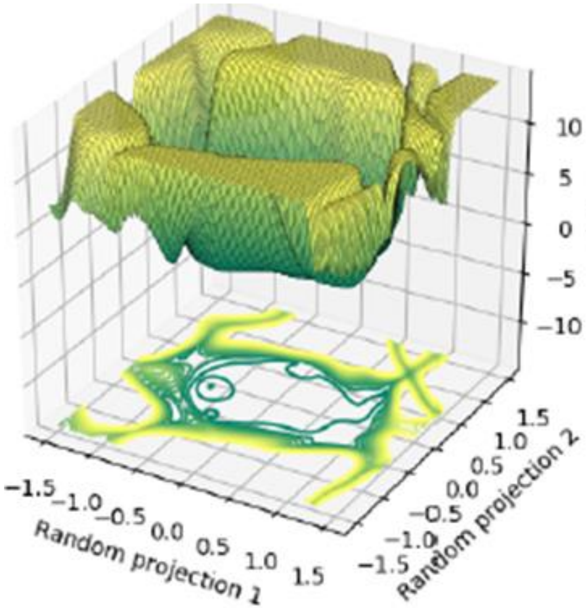
Deep learning involves **complex, compositional, non-linear functions**

The **loss landscape** is extremely **non-convex** as a result

There is **little direct theory** and a **lot of intuition/rules of thumbs** instead

- Some insight can be gained via theory for simpler cases (e.g. convex settings)

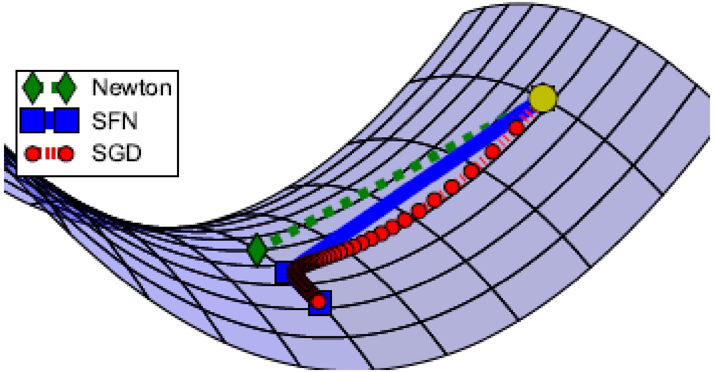


Loss Landscape

It used to be thought that **existence of local minima is the main issue** in optimization

There are other **more impactful issues**:

- Noisy gradient estimates
- Saddle points
- Ill-conditioned loss surface

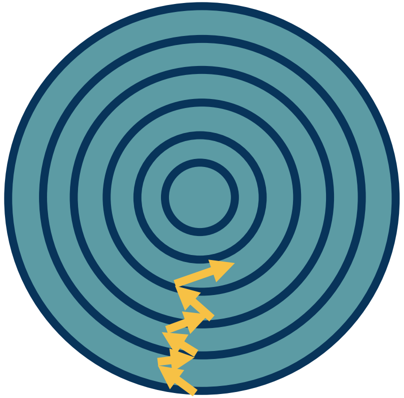


Saddle Point

From: Identifying and attacking the saddle point problem in high-dimensional non-convex optimization, Dauphi et al., 2014.

- We use a **subset of the data at each iteration** to calculate the loss (& gradients)
- This is an **unbiased** estimator but can have high variance
- This results in **noisy steps** in gradient descent

$$L = \frac{1}{M}\sum L\left(f(x_i, W), y_i\right)$$

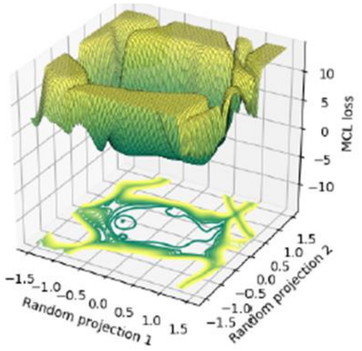


Several **loss surface geometries** are difficult for optimization

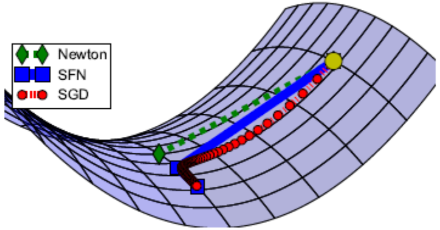
Several types of minima: Local minima, plateaus, saddle points

Saddle points are those where the gradient of orthogonal directions are zero

- But they **disagree** (it's min for one, max for another)



Plateau



Saddle Point

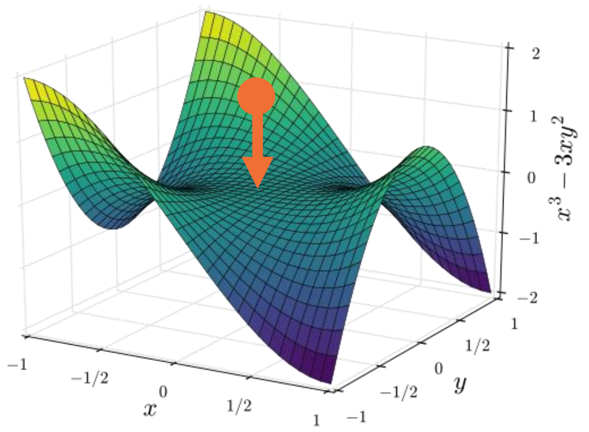
- Gradient descent takes a step in the **steepest direction** (negative gradient)

$$w_i = w_{i-1} - \alpha \frac{\partial L}{\partial w_i}$$

- **Intuitive idea:** Imagine a ball rolling down loss surface, and use **momentum** to pass flat surfaces

$$v_i = \beta v_{i-1} + \frac{\partial L}{\partial w_{i-1}}$$ **Update Velocity (starts as 0, $\beta = 0.99$)**

$$w_i = w_{i-1} - \alpha v_i$$ **Update Weights**



- Generalizes SGD ($\beta = 0$)

- Velocity term is an **exponential moving average** of the gradient

$$v_i = \beta v_{i-1} + \frac{\partial L}{\partial w_{i-1}}$$

$$v_i = \beta(\beta\, v_{i-2} + \frac{\partial L}{\partial w_{i-2}}) + \frac{\partial L}{\partial w_{i-1}}$$

$$= \ \beta^2 v_{i-2} + \beta \frac{\partial L}{\partial w_{i-2}} + \frac{\partial L}{\partial w_{i-1}}$$

- There is a **general class of accelerated gradient methods**, with some theoretical analysis (under assumptions)

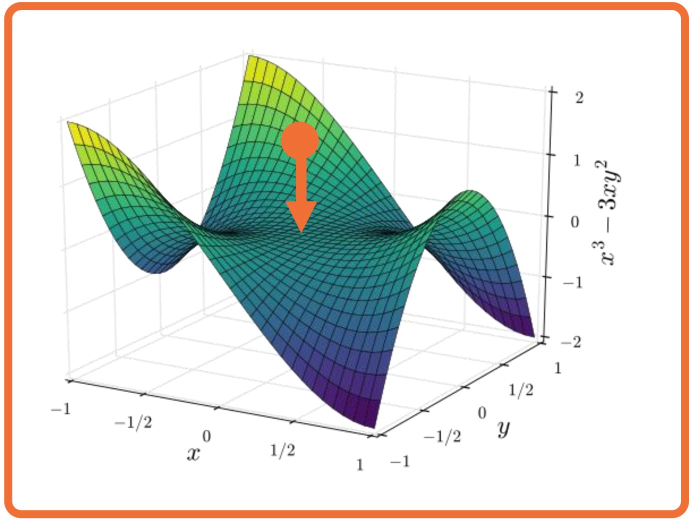
Equivalent formulation:

$$v_i = \beta v_{i-1} - \alpha \frac{\partial L}{\partial w_{i-1}}$$

Update Velocity (starts as 0)

$$w_i = w_{i-1} + v_i$$

Update Weights



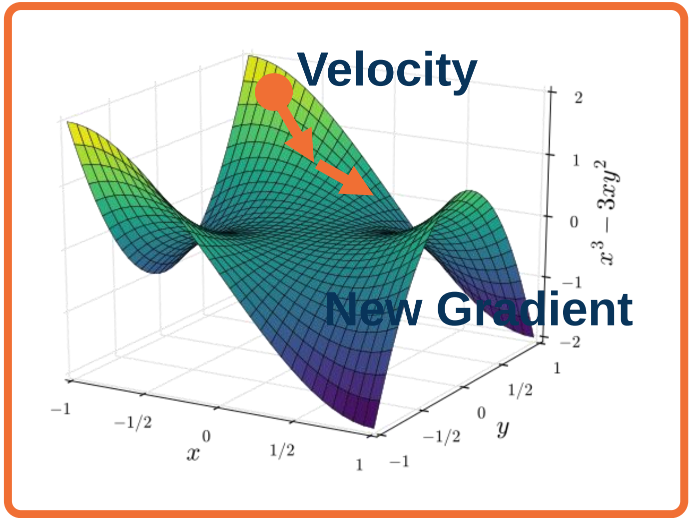
Key idea: Rather than combining velocity with current gradient, go along velocity **first** and then calculate gradient at new point

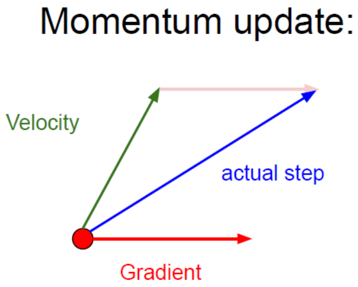
- We know velocity is probably a **reasonable direction**

$$\hat{w}_{i-1} = w_{i-1} + \beta v_{i-1}$$

$$v_i = \beta v_{i-1} + \frac{\partial L}{\partial \hat{w}_{i-1}}$$

$$w_i = w_{i-1} - \alpha\, v_i$$





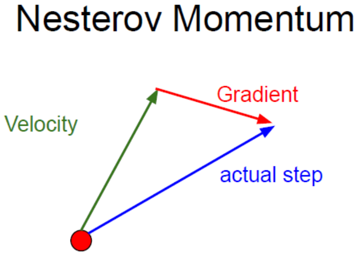


Figure Credit: Fei-Fei Li, Justin Johnson, Serena Yeung, CS 231n

Nesterov Momentum

Momentum

Note there are **several equivalent formulations** across deep learning frameworks!

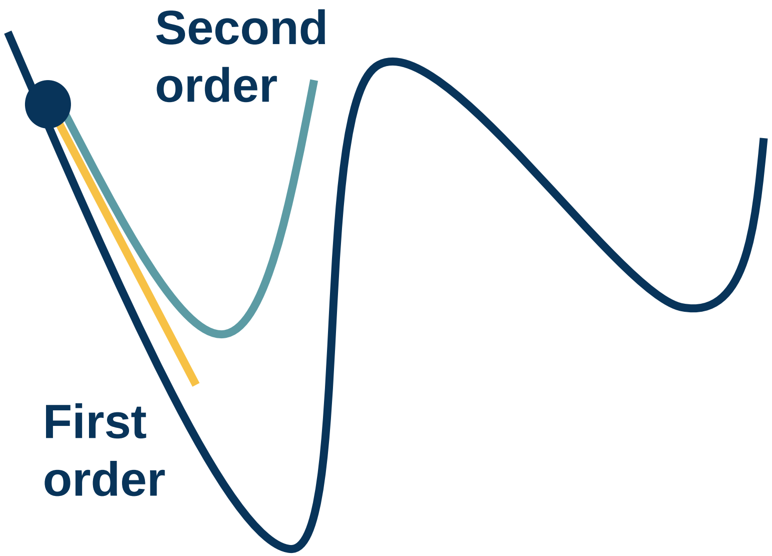
Resource:
https://medium.com/the-artificial-impostor/sgd-implementation-in-pytorch-4115bcb9f02c

- Various mathematical ways to **characterize the loss landscape**

- If you liked **Jacobians**… meet the

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\[2.2ex] \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\[2.2ex] \vdots & \vdots & \ddots & \vdots \\[2.2ex] \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Gives us information about the **curvature of the loss surface**

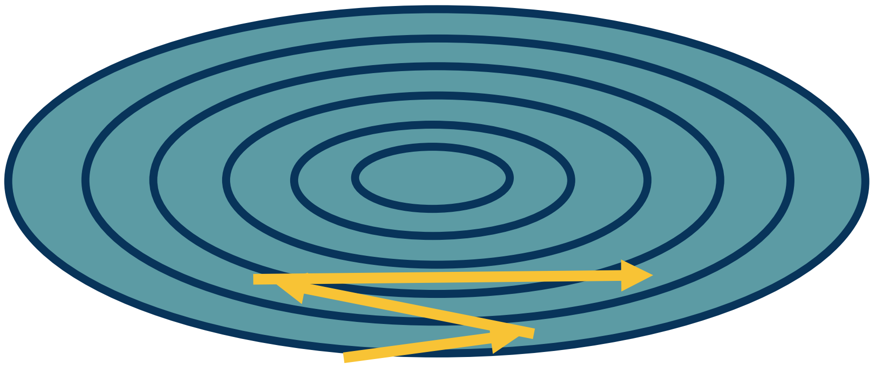


Condition number is the ratio of the largest and smallest eigenvalue

- Tells us how different the curvature is along different dimensions

If this is high, SGD will make **big** steps in some dimensions and **small** steps in other dimension

Second-order optimization methods divide steps by curvature, but expensive to compute



Per-Parameter Learning Rate

Idea: Have a dynamic learning rate for each weight

Several flavors of **optimization algorithms**:

- RMSProp
- Adagrad
- Adam
- …

SGD+Momentum can achieve similar results in many cases but with much more tuning

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Idea: Use gradient statistics to reduce learning rate across iterations

Denominator: Sum up gradients over iterations

Directions with **high curvature will have higher gradients**, and learning rate will reduce

$$G_i = G_{i-1} + \left(\frac{\partial L}{\partial w_{i-1}}\right)^2$$

$$w_i = w_{i-1} - \frac{\alpha}{\sqrt{G_i} + \epsilon}\frac{\partial L}{\partial w_{i-1}}$$

As gradients are accumulated learning rate will go to zero

Duchi, et al., "Adaptive Subgradient Methods for Online Learning and Stochastic Optimization"

Solution: Keep a moving average of squared gradients!

Does not saturate the learning rate

$$G_i = \beta G_{i-1} + (1-\beta)\left(\frac{\partial L}{\partial w_{i-1}}\right)^2$$

$$w_i = w_{i-1} - \frac{\alpha}{\sqrt{G_i + \epsilon}} \frac{\partial L}{\partial w_{i-1}}$$

Combines ideas from above algorithms

Maintains both gradient and squared statistics for gradients

$$v_i = \beta_1 v_{i-1} + (1-\beta_1)\left(\frac{\partial L}{\partial w_{i-1}}\right)$$

$$G_i = \beta_2 G_{i-1} + (1-\beta_2)\left(\frac{\partial L}{\partial w_{i-1}}\right)^2$$

$$w_i = w_{i-1} - \frac{\alpha v_i}{\sqrt{G_i + \epsilon}}$$

But unstable in the beginning (one or both of moments will be tiny values)

Kingma and Ba, "Adam: A method for stochastic optimization", ICLR 2015

Adam

Solution: Time-varying bias correction

Typically $\beta_1 = 0.9,\ \ \beta_2 = 0.999$

So $\hat{v}_i$ will be small number divided by (1-0.9=0.1) resulting in more reasonable values (and $\hat{G}_i$ larger)

$$v_i = \beta_1\, v_{i-1} + (1 - \beta_1)\left(\frac{\partial L}{\partial w_{i-1}}\right)$$

$$G_i = \beta_2\, G_{i-1} + (1 - \beta_2)\left(\frac{\partial L}{\partial w_{i-1}}\right)^2$$

$$\hat{v}_i = \frac{v_i}{1 - \beta_1^i} \qquad \hat{G}_i = \frac{G_i}{1 - \beta_2^i}$$

$$w_i = w_{i-1} - \frac{\alpha\, \hat{v}_i}{\sqrt{\hat{G}_i + \epsilon}}$$

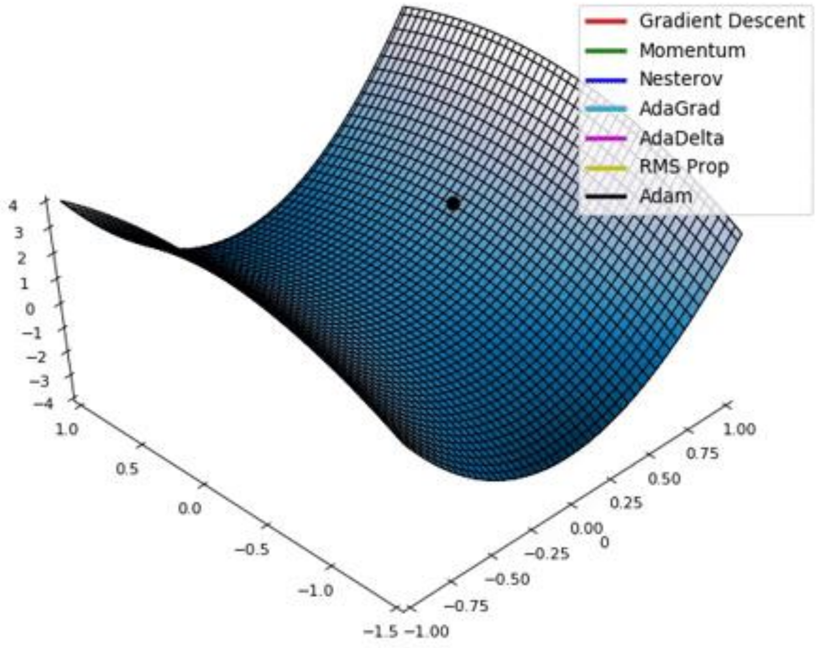
Adam

Optimizers behave differently **depending on landscape**

Different behaviors such as **overshooting, stagnating, etc.**

Plain SGD+Momentum can generalize better than adaptive methods, but requires more tuning

- **See:** *Luo et al., Adaptive Gradient Methods with Dynamic Bound of Learning Rate, ICLR 2019*



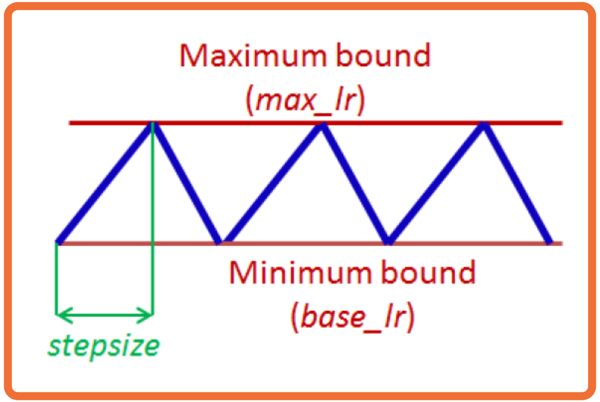
From: https://mlfromscratch.com/optimizers-explained/#/

First order optimization methods have **learning rates**

Theoretical results rely on **annealed learning rate**

Several schedules that are typical:

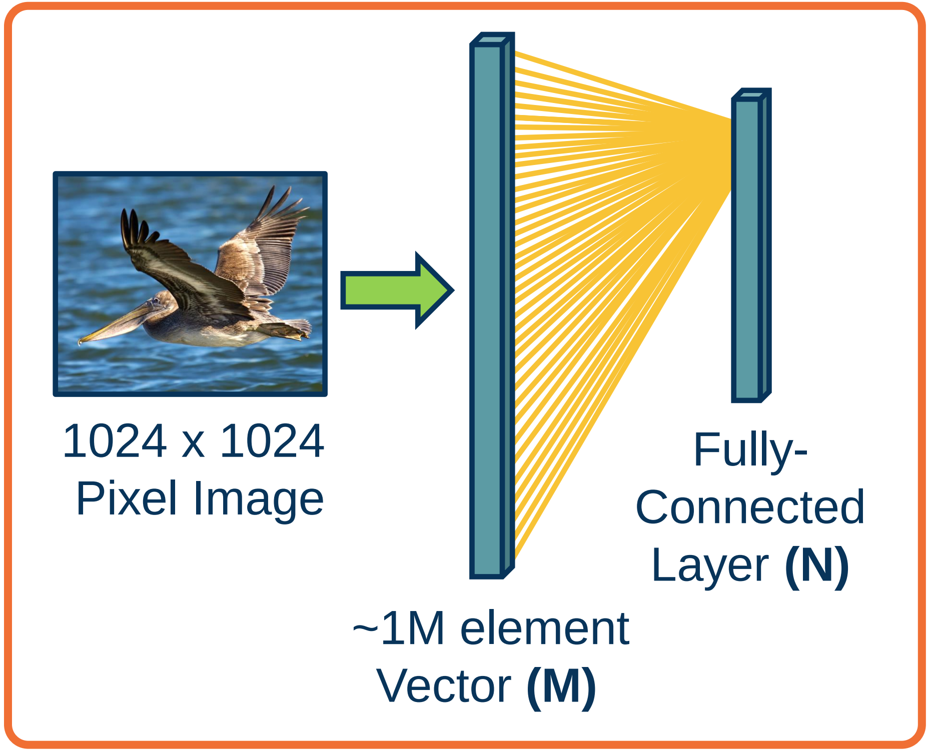
- Graduate student!
- Step scheduler
- Exponential scheduler
- Cosine scheduler



From: Leslie Smith, "Cyclical Learning Rates for Training Neural Networks"

Convolution & Pooling

The connectivity in linear layers **doesn't always make sense**



How many parameters?

- M*N (weights) + N (bias)

Hundreds of millions of parameters **for just one layer**

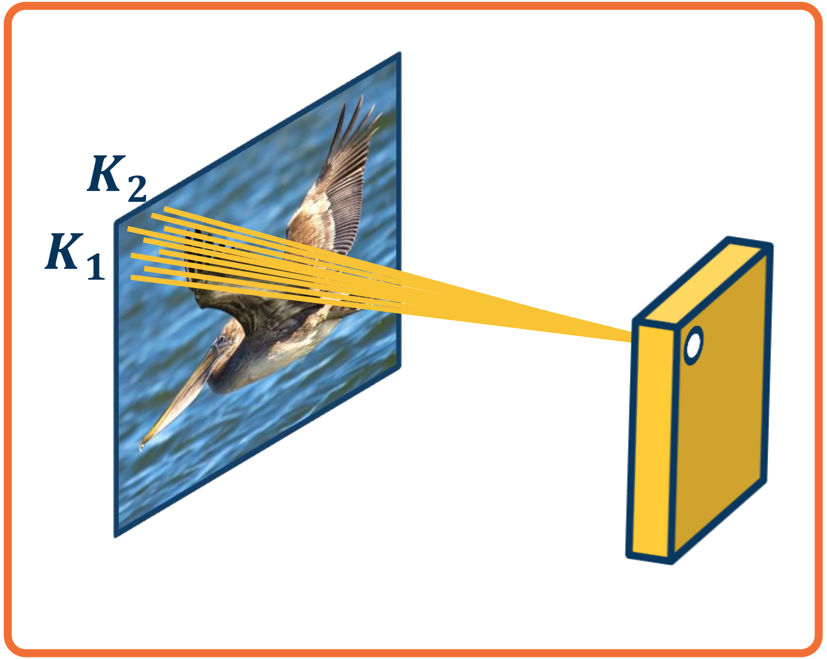
More parameters => More data needed

Is this necessary?

Image features are spatially localized!

- Smaller features repeated across the image
  - Edges
  - Color
  - Motifs (corners, etc.)
- No reason to believe one feature tends to appear in one location vs. another (stationarity)

Can we induce a *bias* in the design of a neural network layer to reflect this?



Each node only receives input from $K_1 \times K_2$ window (image patch)

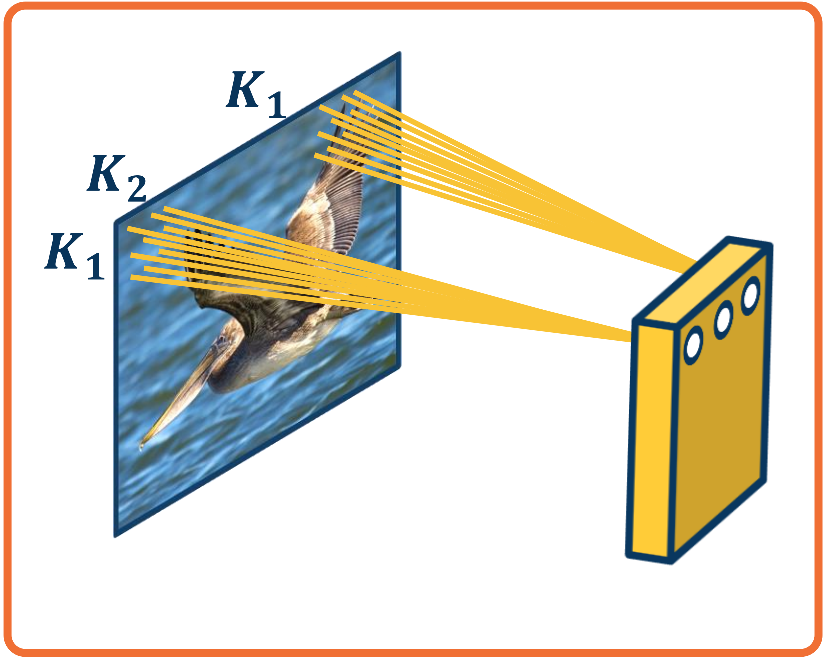
- Region from which a node receives input from is called its **receptive field**

Advantages:

- Reduce parameters to $(K_1 \times K_2 + 1) * N$ where N is number of output nodes
- Explicitly maintain spatial information

Do we need to learn location-specific features?

Idea 1: Receptive Fields



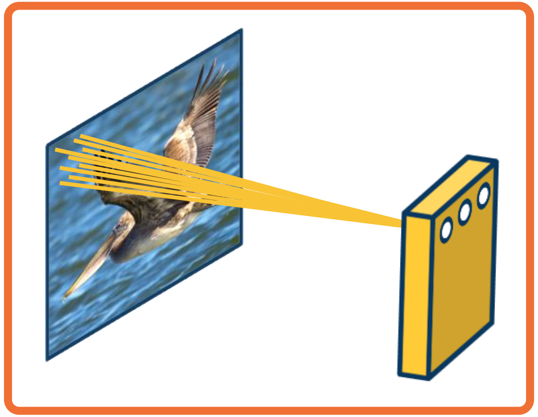
Nodes in different locations can **share** features

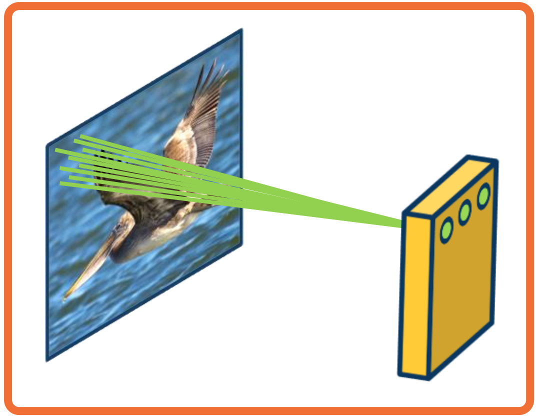
- No reason to think same feature (e.g. edge pattern) can't appear elsewhere
- Use same weights/parameters in computation graph **(shared weights)**

Advantages:

- Reduce parameters to $(\boldsymbol{K_1} \times \boldsymbol{K_2} + \mathbf{1})$
- Explicitly maintain spatial information

Idea 2: Shared Weights



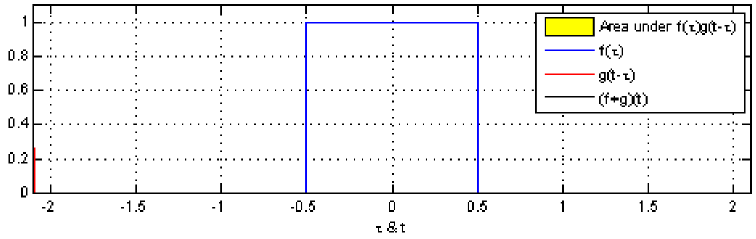


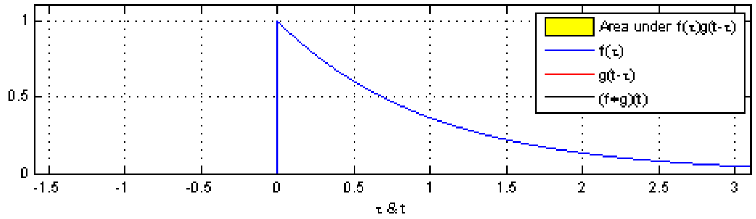
We can learn **many** such features for this one layer

- Weights are **not** shared across different feature extractors
- **Parameters:** $(K_1 \times K_2 + 1) * M$ where M is number of features we want to learn

Idea 3: Learn Many Features

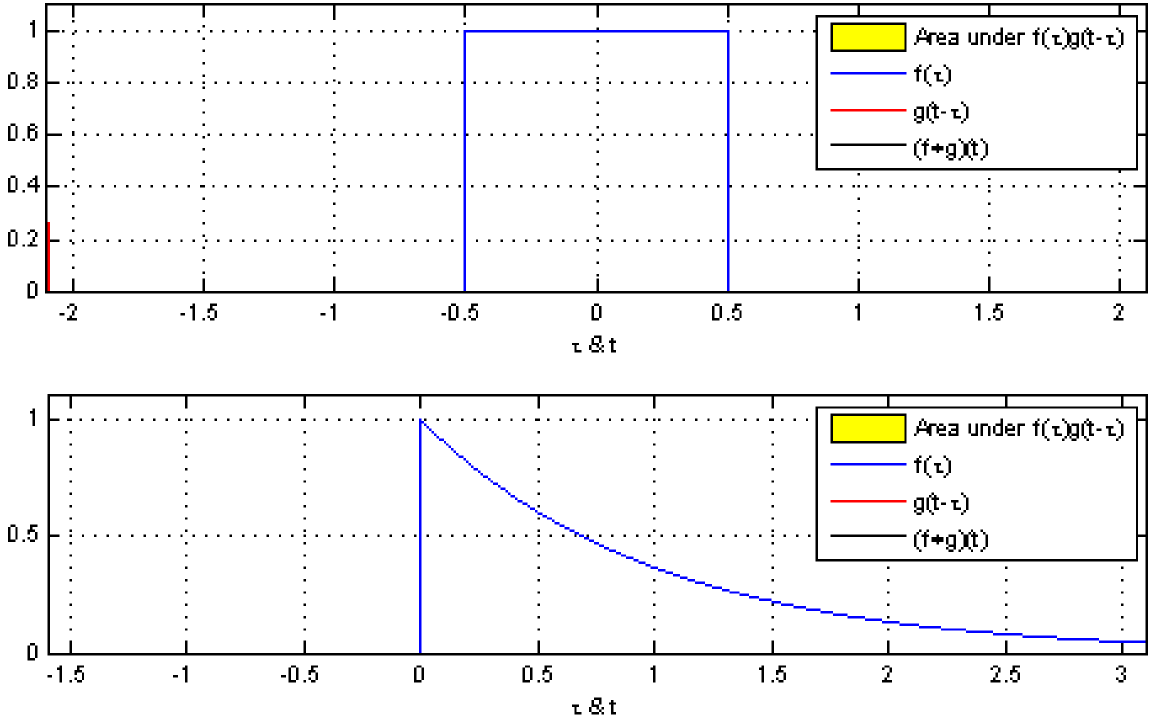
This operation is **extremely common** in electrical/computer engineering!





From https://en.wikipedia.org/wiki/Convolution

This operation is **extremely common** in electrical/computer engineering!



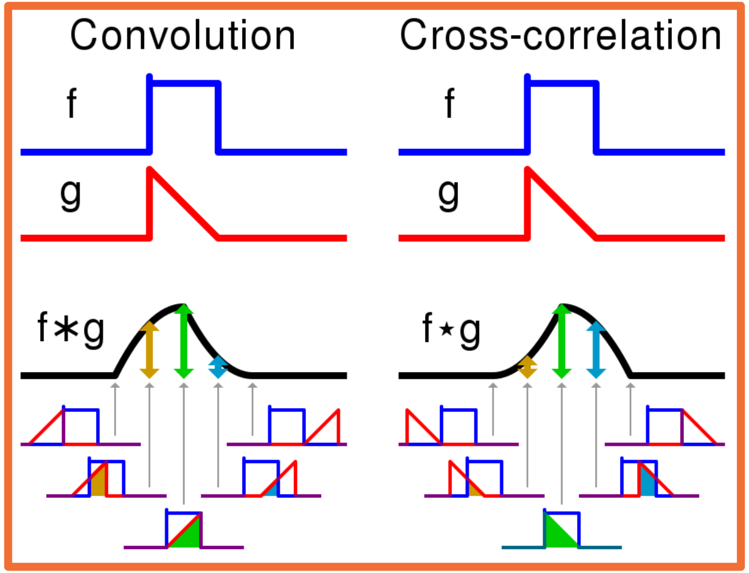
From https://en.wikipedia.org/wiki/Convolution

This operation is **extremely common** in electrical/computer engineering!

In mathematics and, in particular, functional analysis, **convolution** is a mathematical operation on two functions f and g producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated.

Convolution is similar to **cross-correlation**.

It has **applications** that include probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations.



Visual comparison of **convolution** and **cross-correlation.**

From https://en.wikipedia.org/wiki/Convolution

Notation: $F \otimes (G \otimes I) = (F \otimes G) \otimes I$

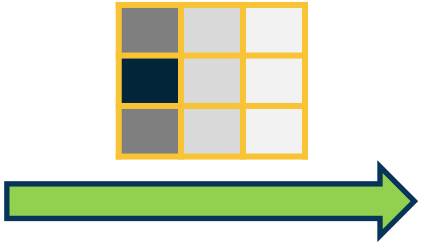
1D Convolution

$$y_k = \sum_{n=0}^{N-1} h_n \cdot x_{k-n}$$

$$y_0 = h_0 \cdot x_0$$
$$y_1 = h_1 \cdot x_0 + h_0 \cdot x_1$$
$$y_2 = h_2 \cdot x_0 + h_1 \cdot x_1 + h_0 \cdot x_2$$
$$y_3 = h_3 \cdot x_0 + h_2 \cdot x_1 + h_1 \cdot x_2 + h_0 \cdot x_3$$
$$\vdots$$

2D Convolution

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



2D Discrete Convolution

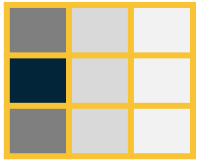
2D Discrete Convolution

2D Convolution

Image

Kernel
(or filter)

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



Output /
filter /
feature map

We will make this convolution operation **a layer** in the neural network

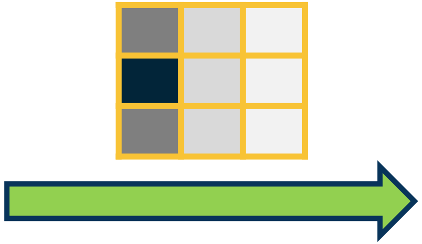
- Initialize kernel values randomly and optimize them!
- These are our parameters (plus a bias term per filter)

2D Convolution

Image

Kernel
(or filter)

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



Output /
filter /
feature map

- **Convolution:** Start at end of kernel and move back
- **Cross-correlation:** Start in the beginning of kernel and move forward (same as for image)

An **intuitive interpretation** of the relationship:

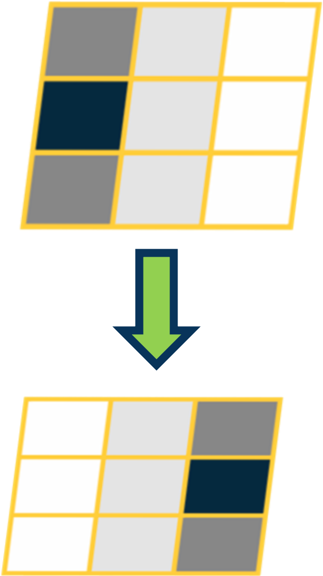
- Take the kernel, and rotate 180 degrees along center (sometimes referred to as “flip”)
- Perform cross-correlation
- (Just dot-product filter with image!)

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

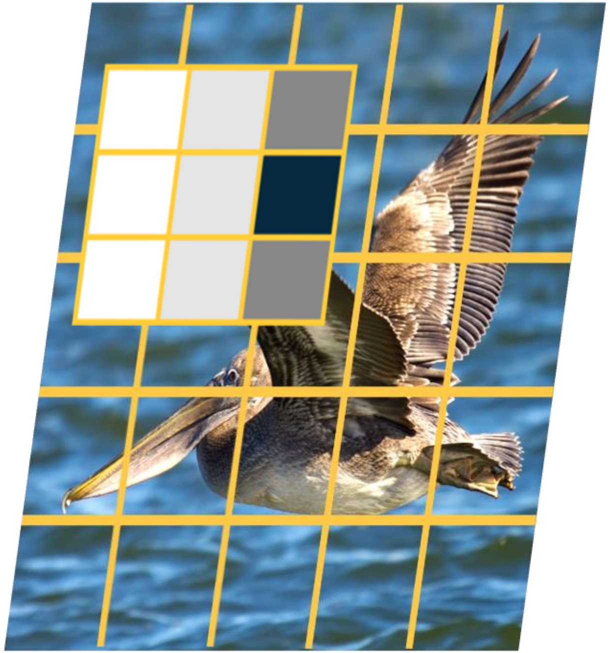
$$\Downarrow$$

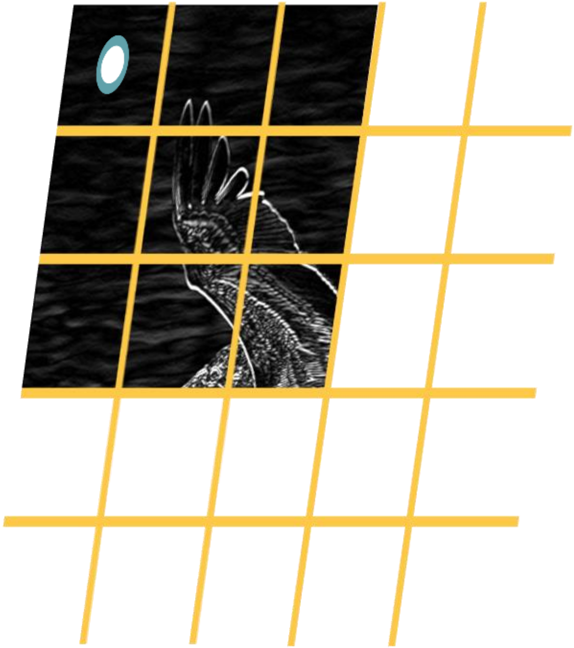
$$K' = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

1. Flip kernel (rotate 180 degrees)

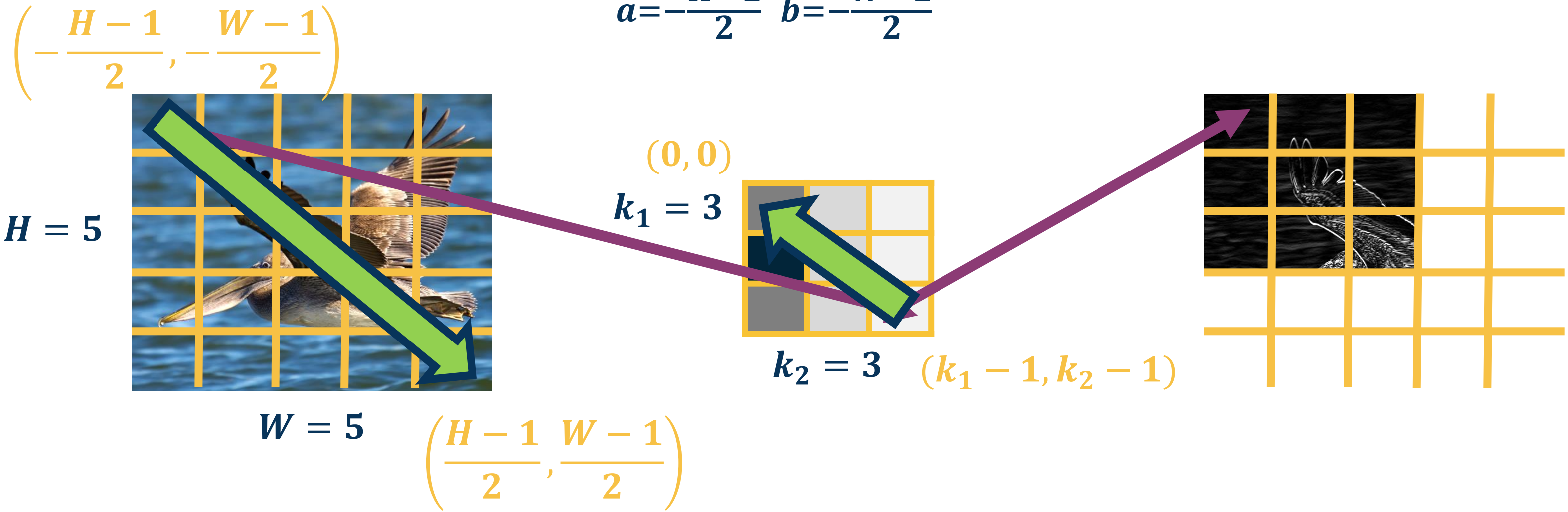


2. Stride along image





$$y(r,c) = (x * k)(r,c) = \sum_{a=-\frac{H-1}{2}}^{\frac{H-1}{2}} \sum_{b=-\frac{W-1}{2}}^{\frac{W-1}{2}} x(a,b)\, k(r-a, c-b)$$



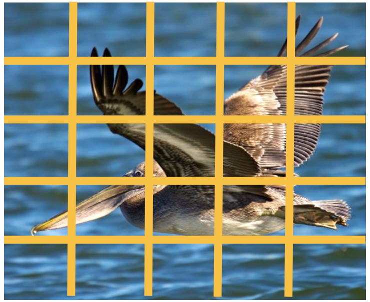
$$y(0,0) = x(-2,-2)k(2,2) + x(-2,-1)k(2,1) + x(-2,0)k(2,0) + x(-2,1)k(2,-1) + x(-2,2)k(2,-2) + \ldots$$

Mathematics of Discrete 2D Convolution

$$y(r,c) = (x * k)(r,c) \;=\; \sum_{a=-\frac{K_1-1}{2}}^{\frac{k_1-1}{2},} \; \sum_{b=-\frac{k_2-1}{2}}^{\frac{k_2-1}{2},} x(r-a,c-b)\, k(a,b)$$

$(0,0)$

$H = 5$

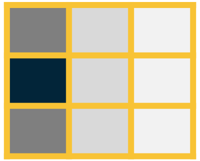


$W = 5$

$(H-1, W-1)$

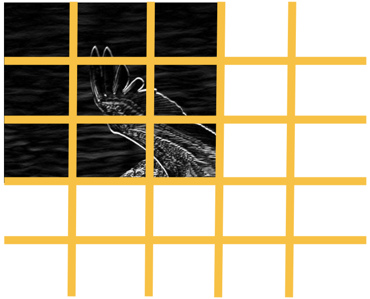
$\left(-\frac{k_1-1}{2}, -\frac{k_2-1}{2}\right)$

$k_1 = 3$



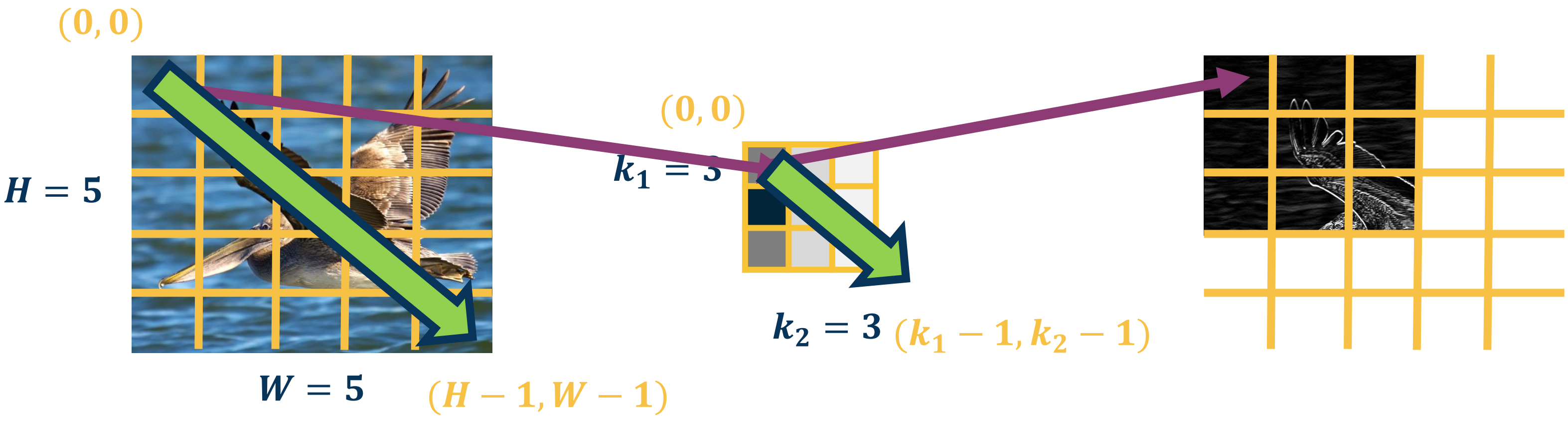
$k_2 = 3$

$\left(\frac{k_1-1}{2}, \frac{k_2-1}{2}\right)$



Centering Around the Kernel

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$

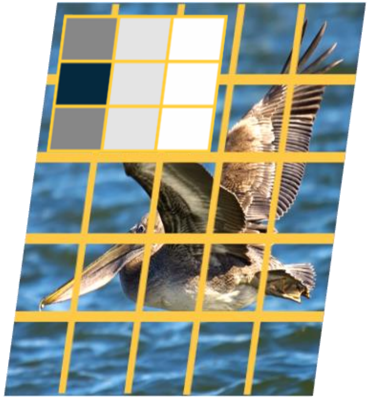


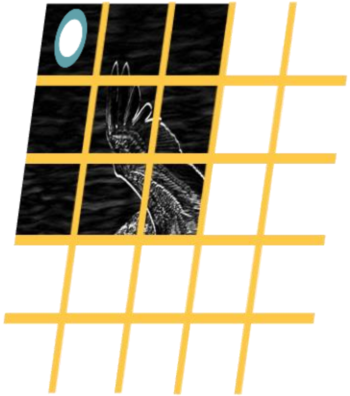
Since we will be learning these kernels, this change does not matter!

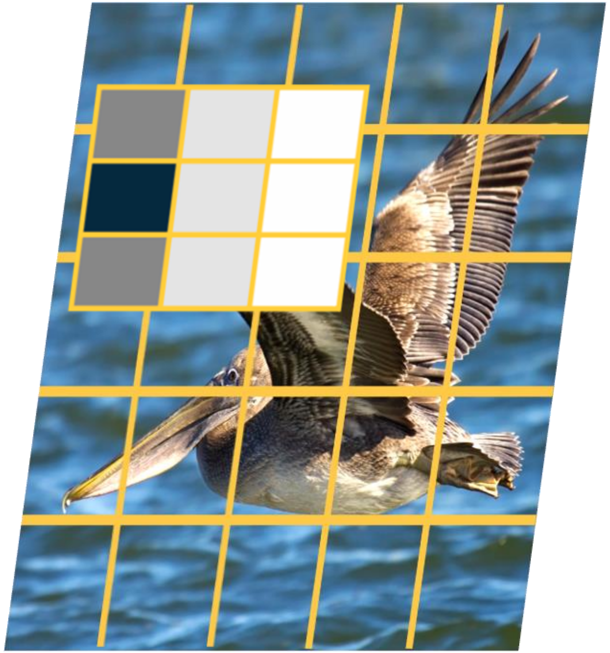
Cross-Correlation

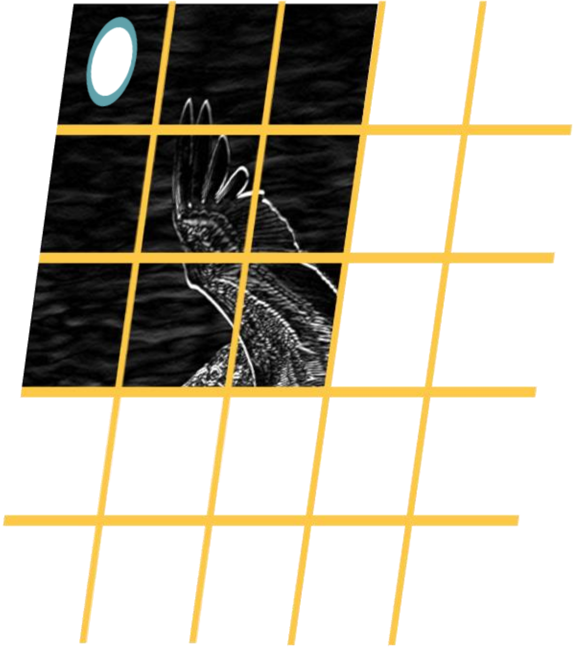
$$X(0{:}2,0{:}2) = \begin{bmatrix} 200 & 150 & 150 \\ 100 & 50 & 100 \\ 25 & 25 & 10 \end{bmatrix} \qquad K' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow X(0{:}2,0{:}2) \cdot K' = 65 + \text{bias}$$

Dot product
(element-wise multiply and sum)

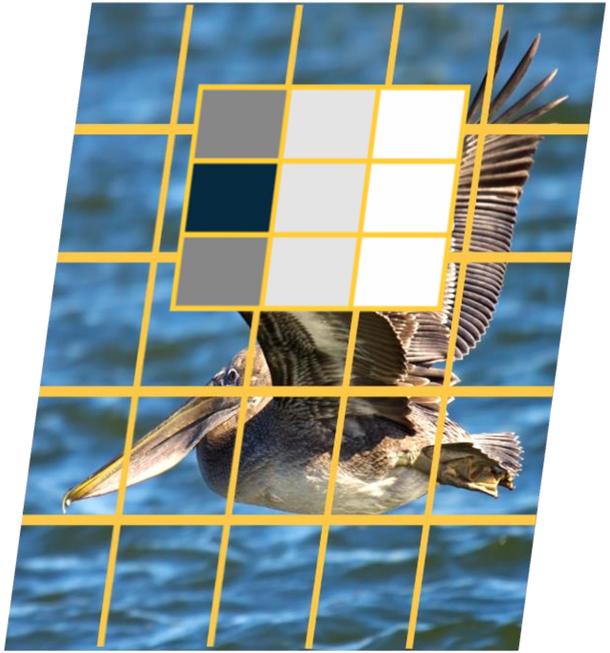


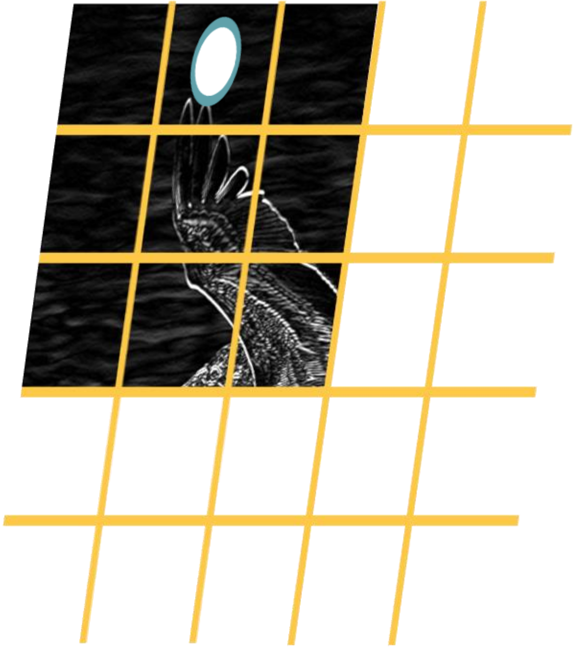




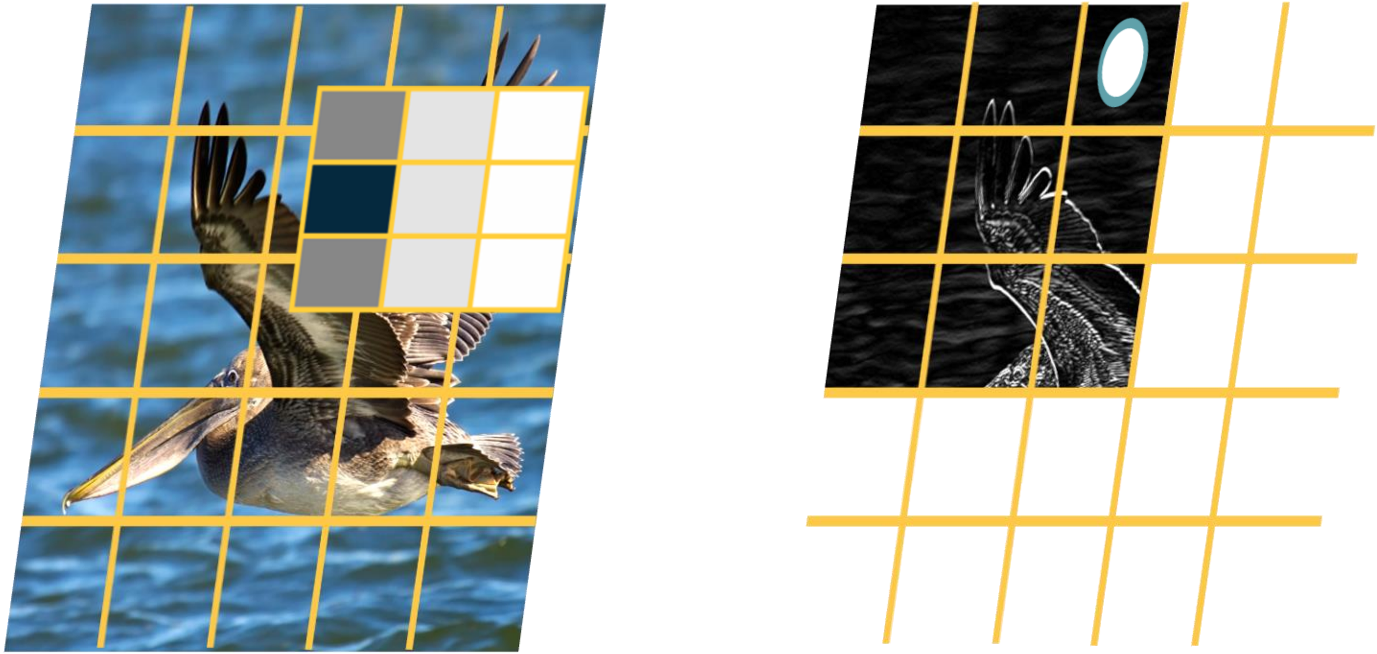


Convolution and Cross-Correlation

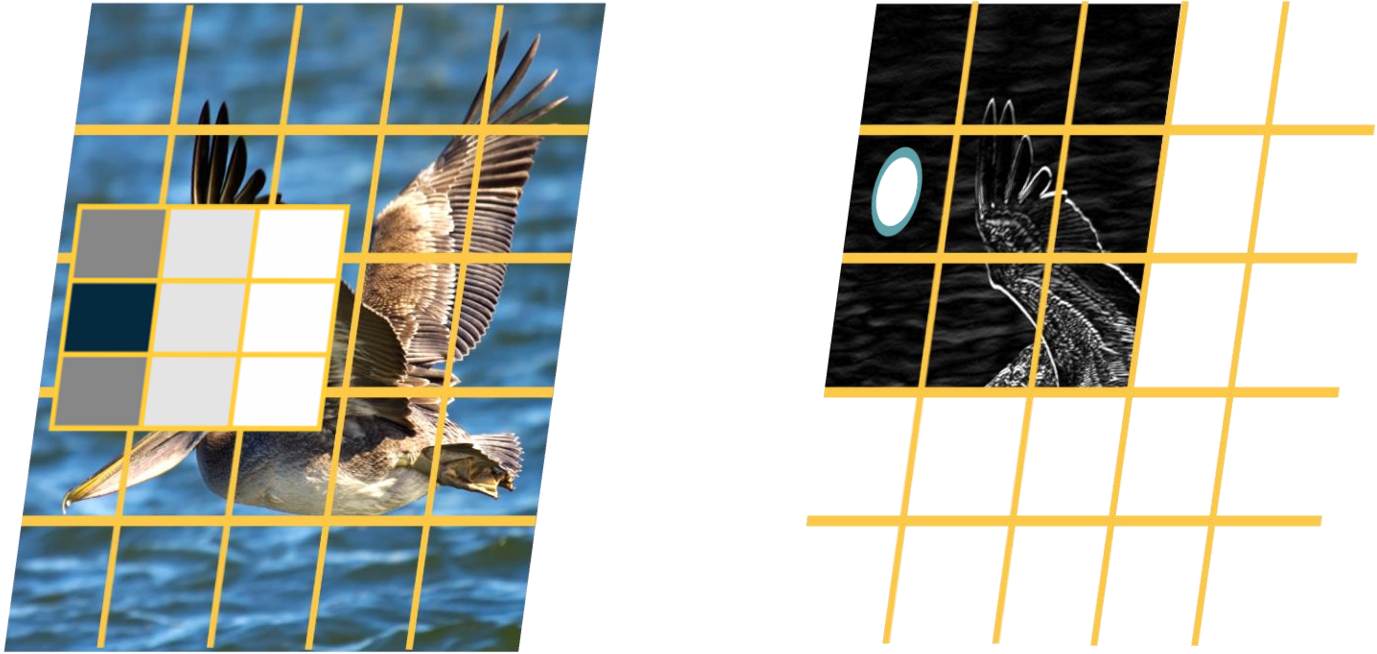




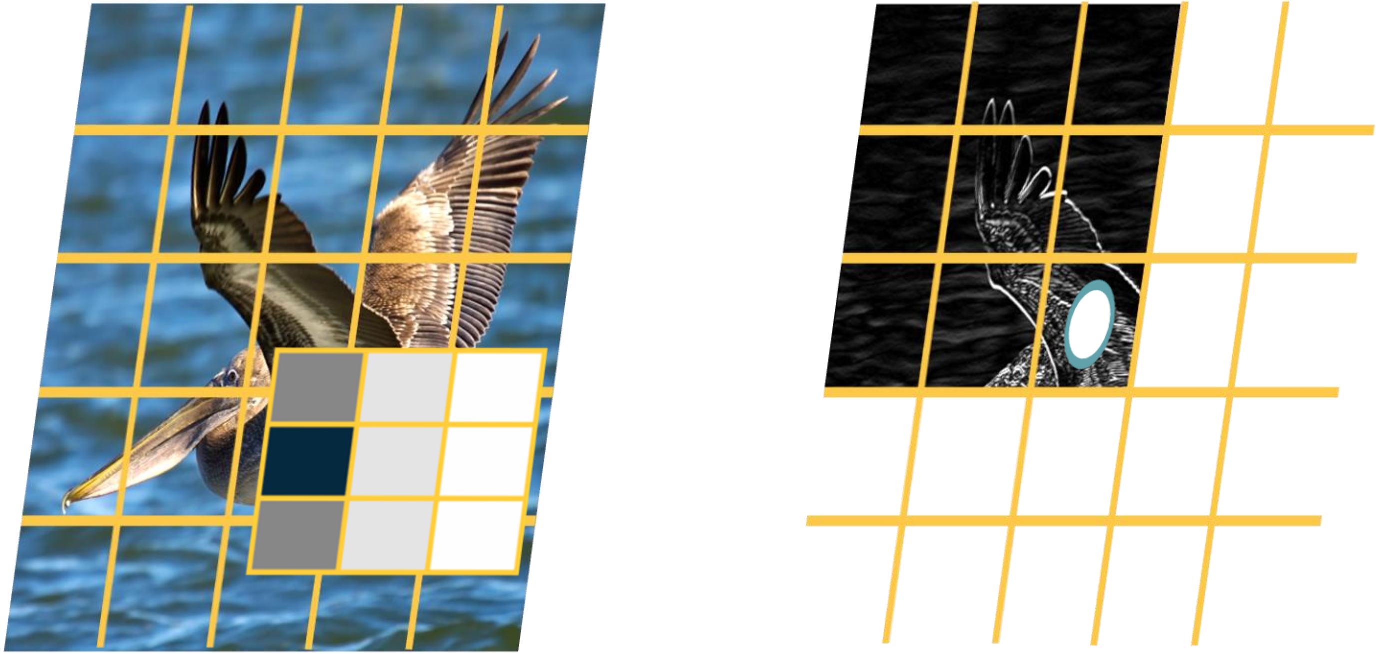
Convolution and Cross-Correlation



Convolution and Cross-Correlation



Convolution and Cross-Correlation



Convolution and Cross-Correlation

Why Bother with Convolutions?

Convolutions are just **simple linear operations**

Why bother with this and not just say it's a linear layer with small receptive field?

- There is a **duality** between them during backpropagation
- Convolutions have **various mathematical properties** people care about
- This is **historically** how it was inspired

Input & Output Sizes

Convolution Layer Hyper-Parameters

Parameters

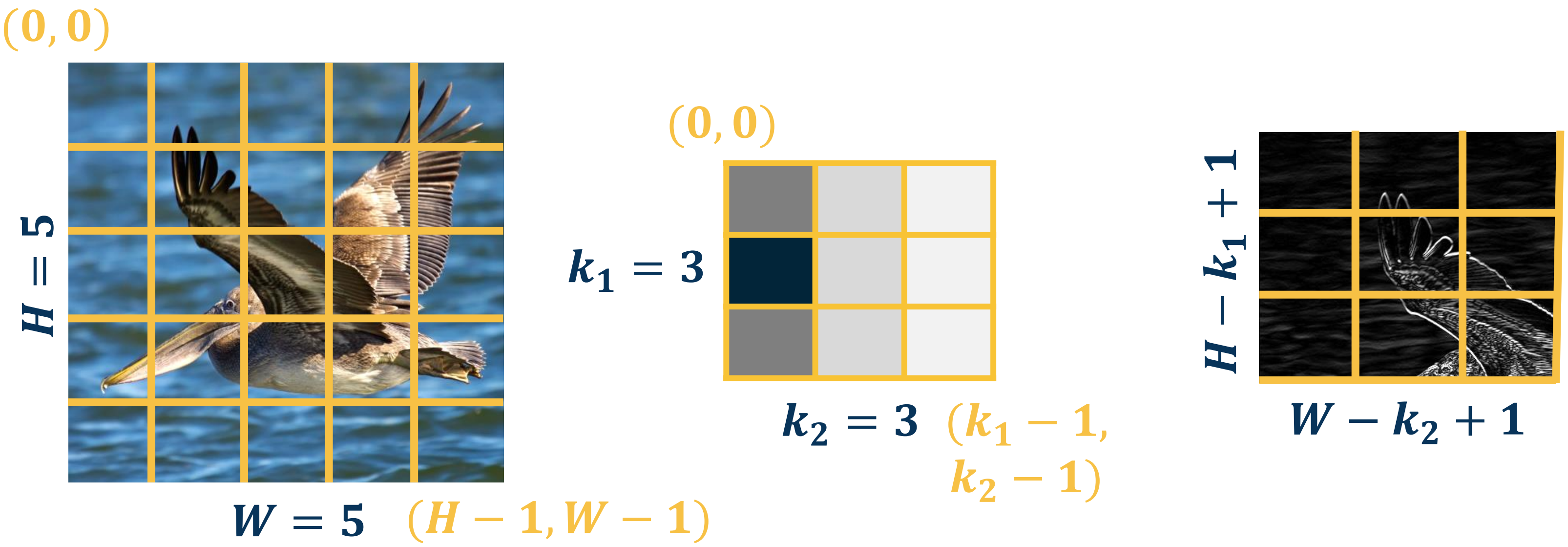
- **in_channels** (*int*) – Number of channels in the input image
- **out_channels** (*int*) – Number of channels produced by the convolution
- **kernel_size** (*int or tuple*) – Size of the convolving kernel
- **stride** (*int or tuple, optional*) – Stride of the convolution. Default: 1
- **padding** (*int or tuple, optional*) – Zero-padding added to both sides of the input. Default: 0
- **padding_mode** (*string, optional*) – `'zeros'`, `'reflect'`, `'replicate'` or `'circular'`. Default: `'zeros'`

Convolution operations have several hyper-parameters

From: https://pytorch.org/docs/stable/generated/torch.nn.Conv2d.html#torch.nn.Conv2d

Output size of vanilla convolution operation is $(H - k_1 + 1) \times (W - k_2 + 1)$

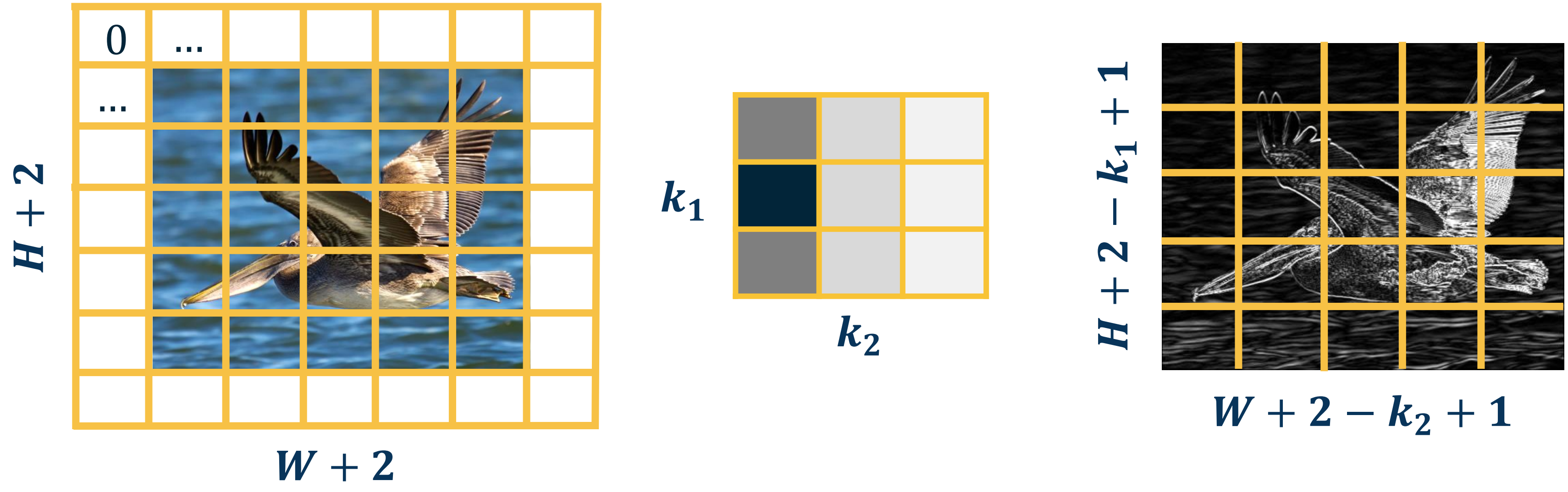
- This is called a **"valid" convolution** and only applies kernel within image



Valid Convolution

We can **pad the images** to make the output the same size:

- Zeros, mirrored image, etc.
- Note padding often refers to pixels added to **one size $(\mathbf{P} = \mathbf{1}\ \mathbf{here})$**



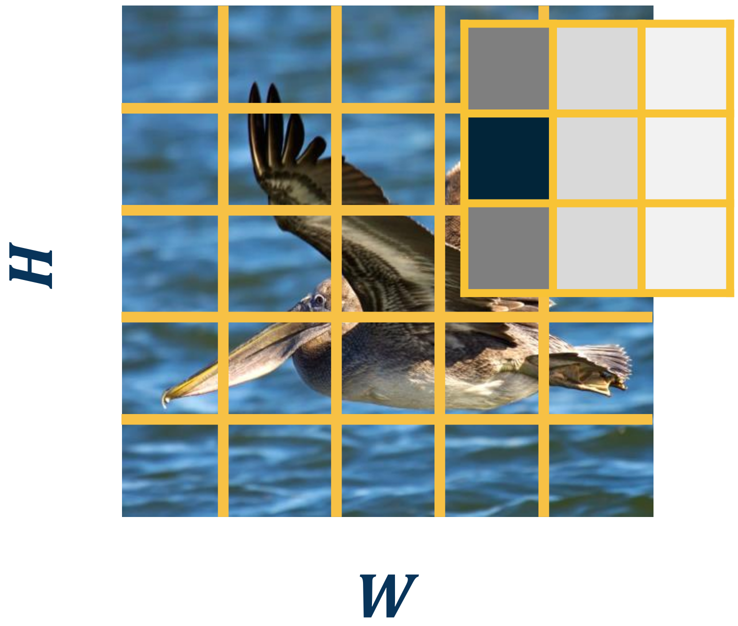
Adding Padding

We can move the filter along the image using larger steps **(stride)**

- This can potentially result in **loss of information**
- Can be used for **dimensionality reduction** (not recommended)

Stride = 2 (every other pixel)

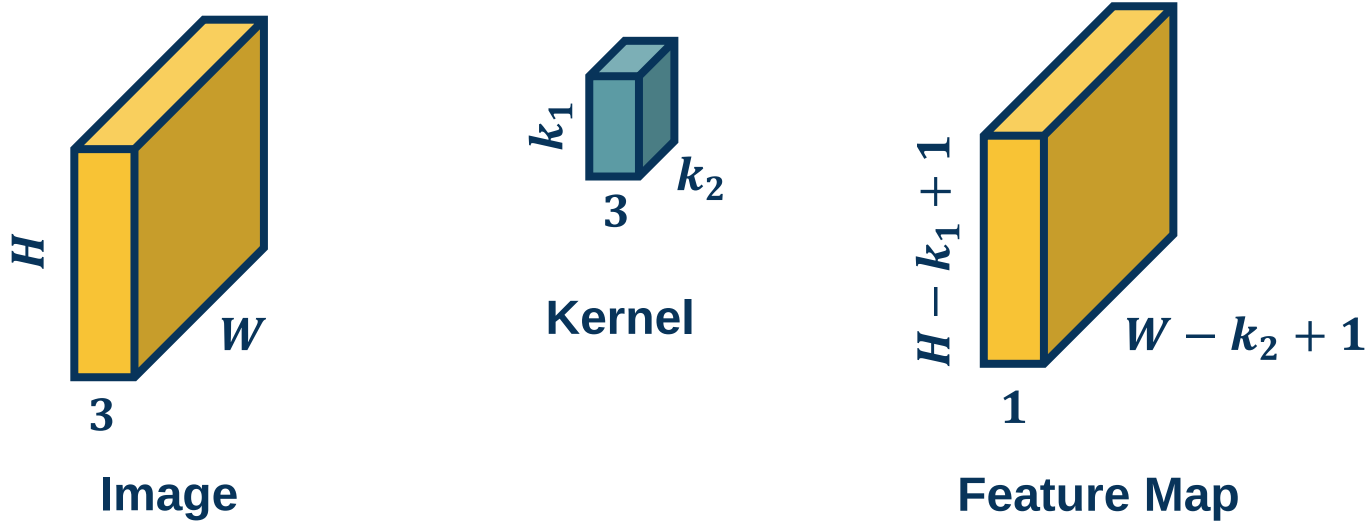
Stride can result in **skipped pixels**, e.g. stride of 3 for 5x5 input



Invalid Stride

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

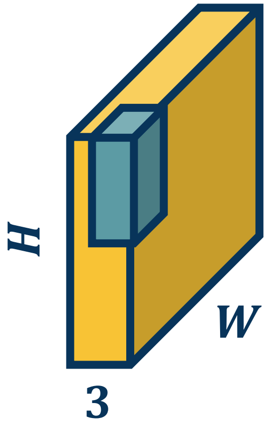
- In such cases, we have **3-channel kernels**!



Multi-Channel Inputs

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

- In such cases, we have **3-channel kernels**!



Image

Similar to before, we perform **element-wise multiplication** between kernel and image patch, summing them up **(dot product)**

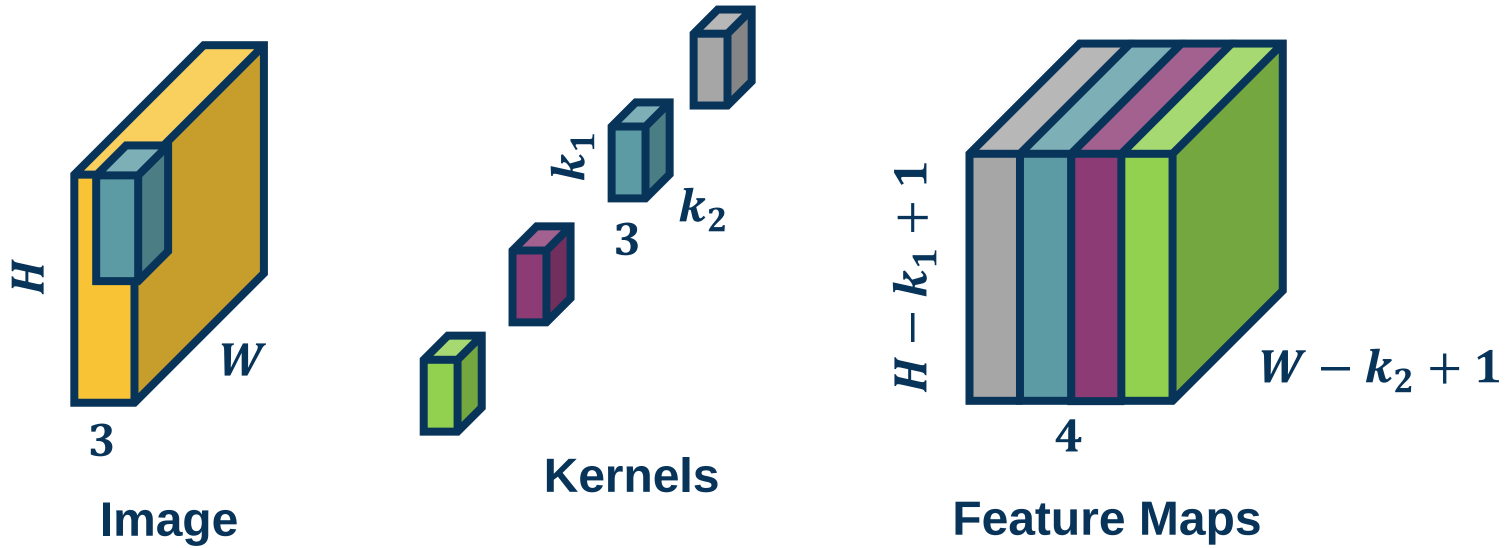
- Except with $\boldsymbol{k_1} * \boldsymbol{k_2} * \mathbf{3}$ values

Operation of Multi-Channel Input

We can have **multiple kernels per layer**

- We stack the feature maps together at the output

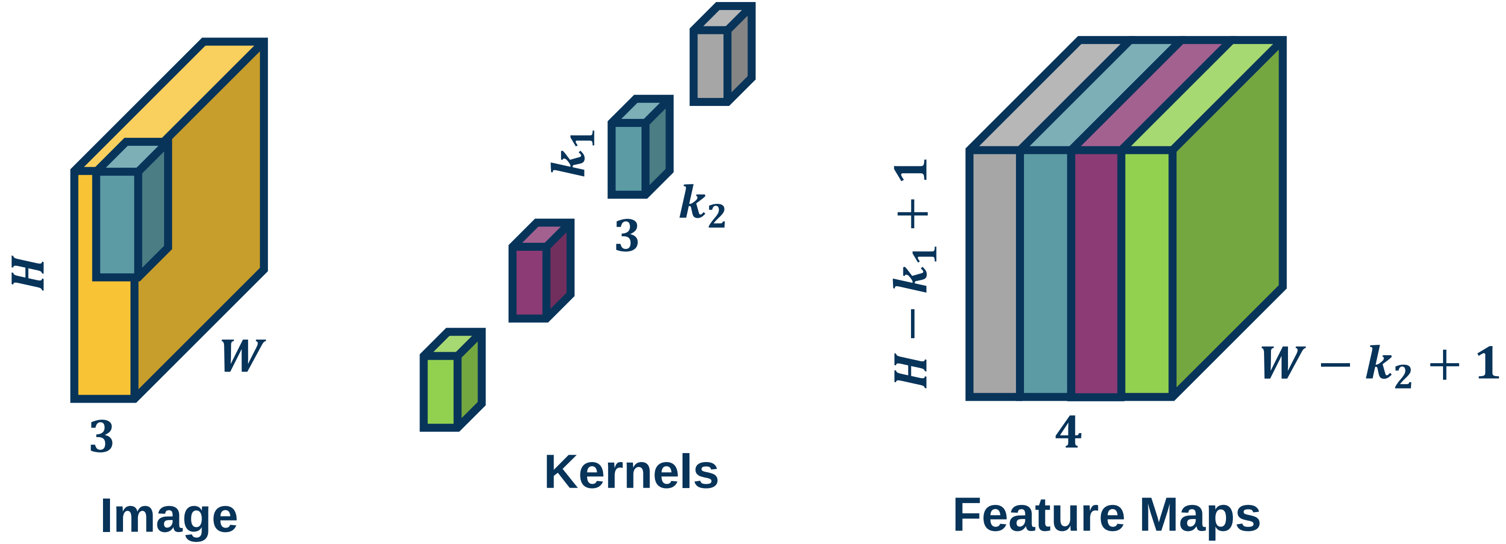
Number of channels in output is equal to *number of kernels*



Multiple Kernels

Number of parameters with N filters is: $\boldsymbol{N * (k_1 * k_2 * 3 + 1)}$

- Example:
  $\boldsymbol{k_1 = 3, k_2 = 3, \ N = 4 \ input\ channels = 3}$, then $\boldsymbol{(3 * 3 * 3 + 1) * 4 = 112}$

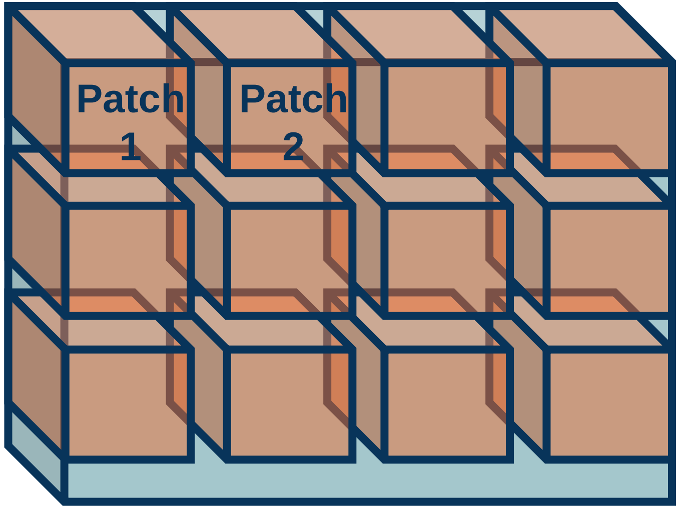


Number of Parameters

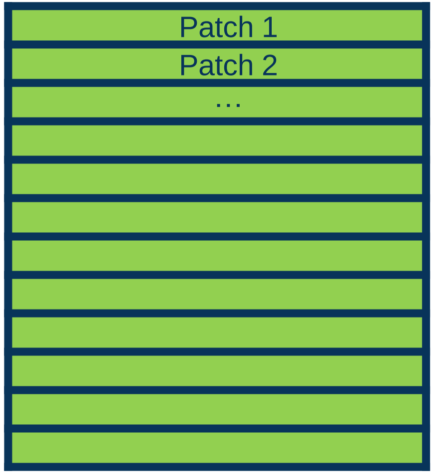
Just as before, in practice we can **vectorize** this operation

- **Step 1:** Lay out image patches in vector form (note can overlap!)

Input Image



Im2col
=>

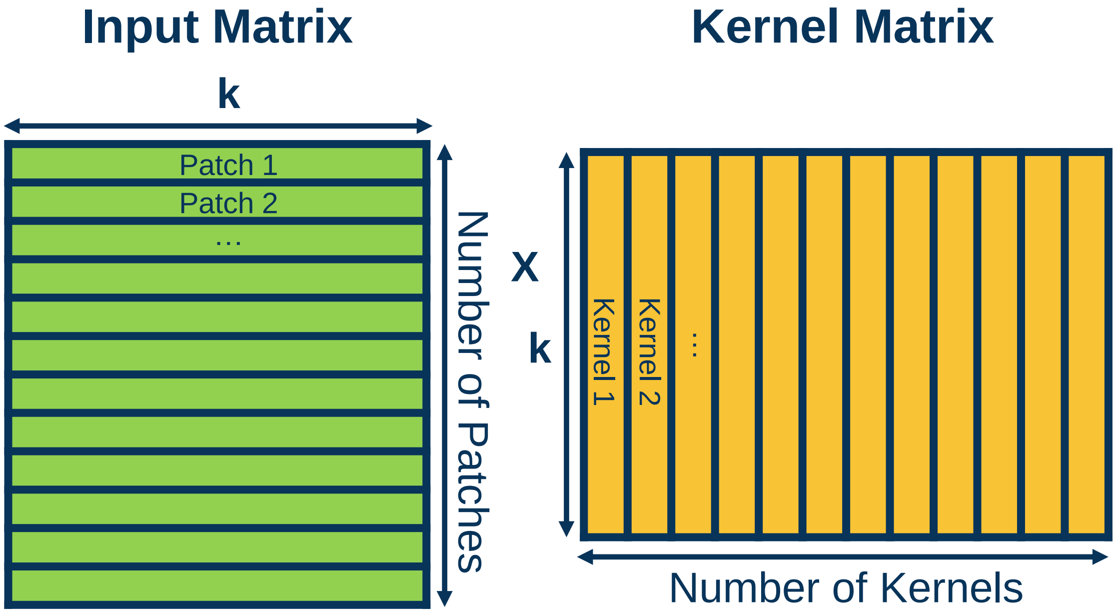


Adapted from: https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/

Vectorization

Just as before, in practice we can **vectorize** this operation

- **Step 2**: Multiple patches by kernels



Adapted from: https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/

Vectorization

- We will have a new layer: Convolution layer
  - Mathematical way of representing a strided filter
    - Equivalent view: Each output node is connected to window, not all input pixels
  - Kernels/filters/features are learned
  - Implementation is actually cross-correlation! (but it doesn't matter)

- Next time: How do we compute the gradients across this layer?
  - Need to reason about what input/weight pixel is affecting what output pixel!

Summary