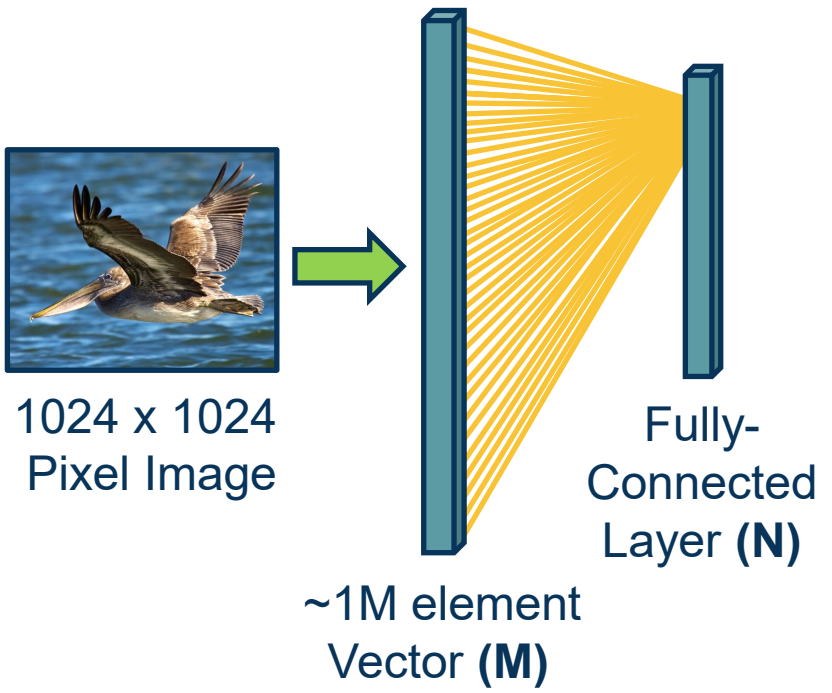
Topics:

- Convolutional Neural Networks

CS 4644-DL / 7643-A
ZSOLT KIRA

The connectivity in linear layers **doesn't always make sense**



How many parameters?

- M*N (weights) + N (bias)

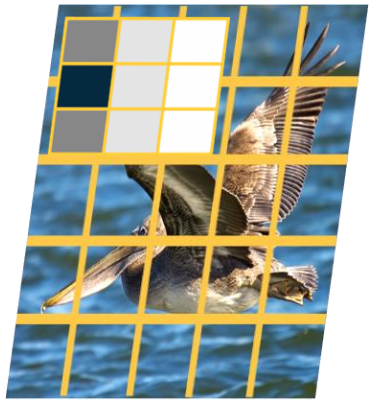
Hundreds of millions of parameters **for just one layer**

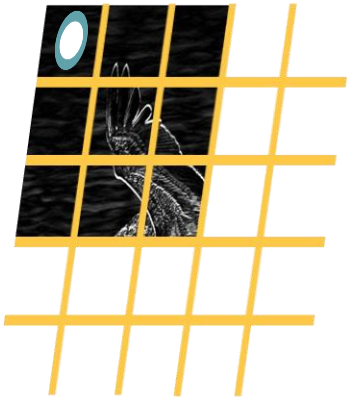
More parameters => More data needed

Is this necessary?

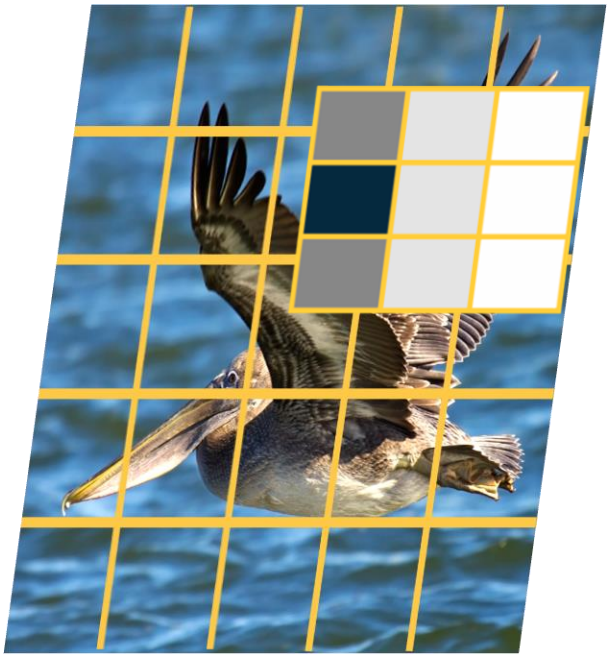
$$X(0:2,0:2) = \begin{bmatrix} 200 & 150 & 150 \\ 100 & 50 & 100 \\ 25 & 25 & 10 \end{bmatrix} \qquad K' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow X(0:2,0:2) \cdot K' = 65 + \text{bias}$$

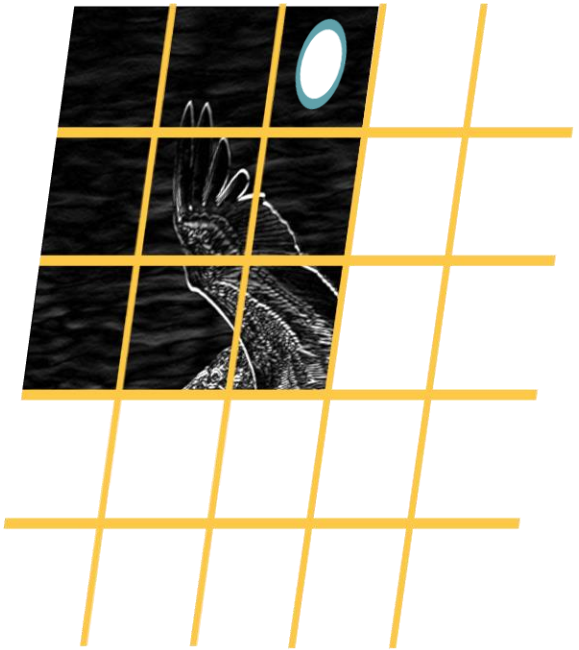
Dot product
(element-wise multiply and sum)



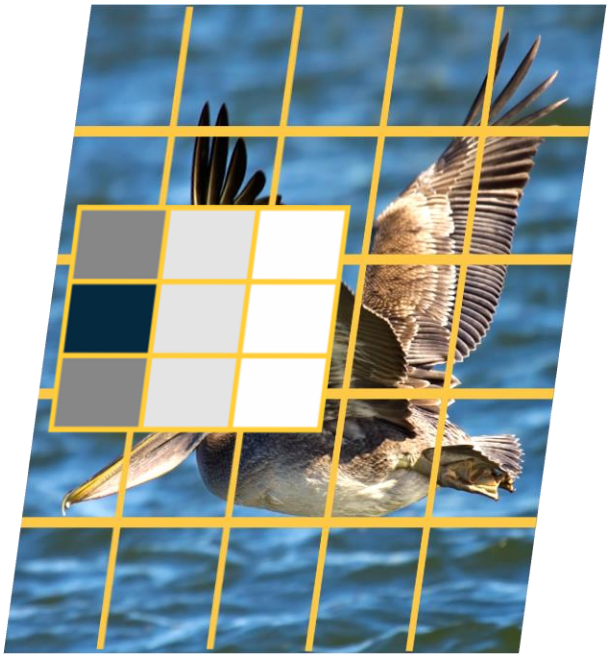


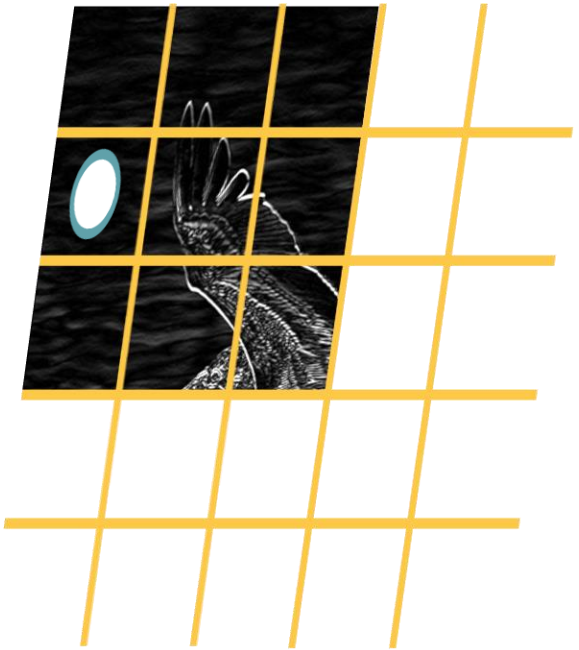
Cross-Correlation



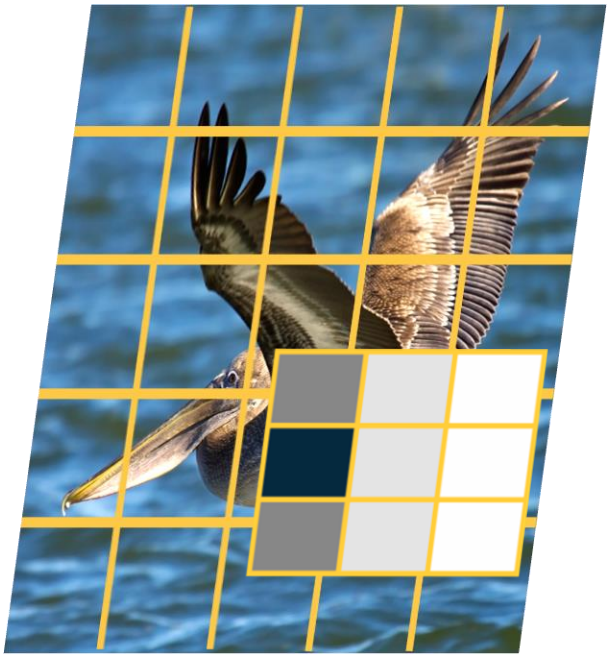


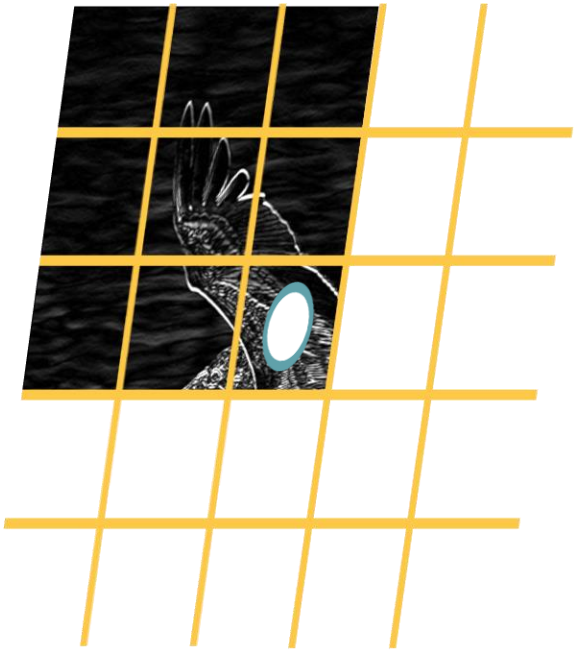
Convolution and Cross-Correlation





Convolution and Cross-Correlation





Convolution and Cross-Correlation

Why Bother with Convolutions?

Convolutions are just **simple linear operations**

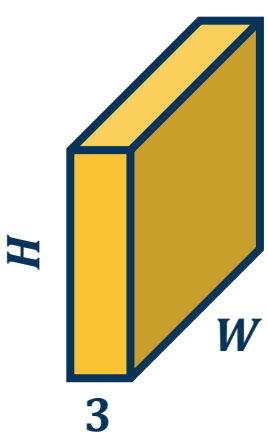
Why bother with this and not just say it's a linear layer with small receptive field?

- There is a **duality** between them during backpropagation
- Convolutions have **various mathematical properties** people care about
- This is **historically** how it was inspired

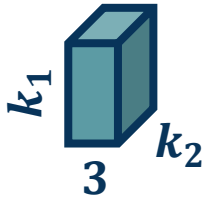
Georgia Tech

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

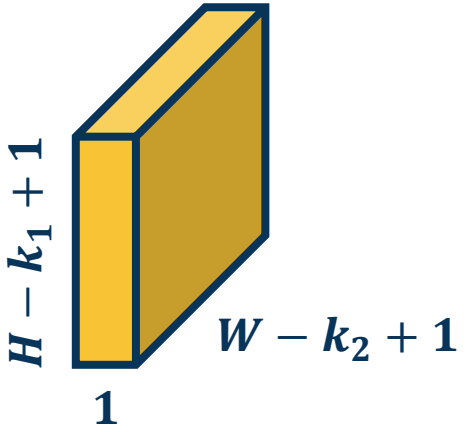
- In such cases, we have **3-channel kernels**!



Image



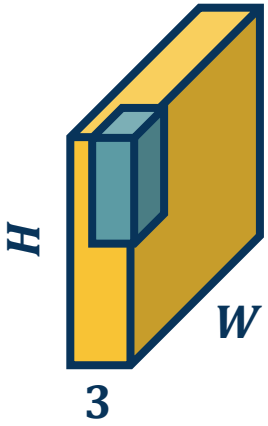
Kernel



Feature Map

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

- In such cases, we have **3-channel kernels**!



Image

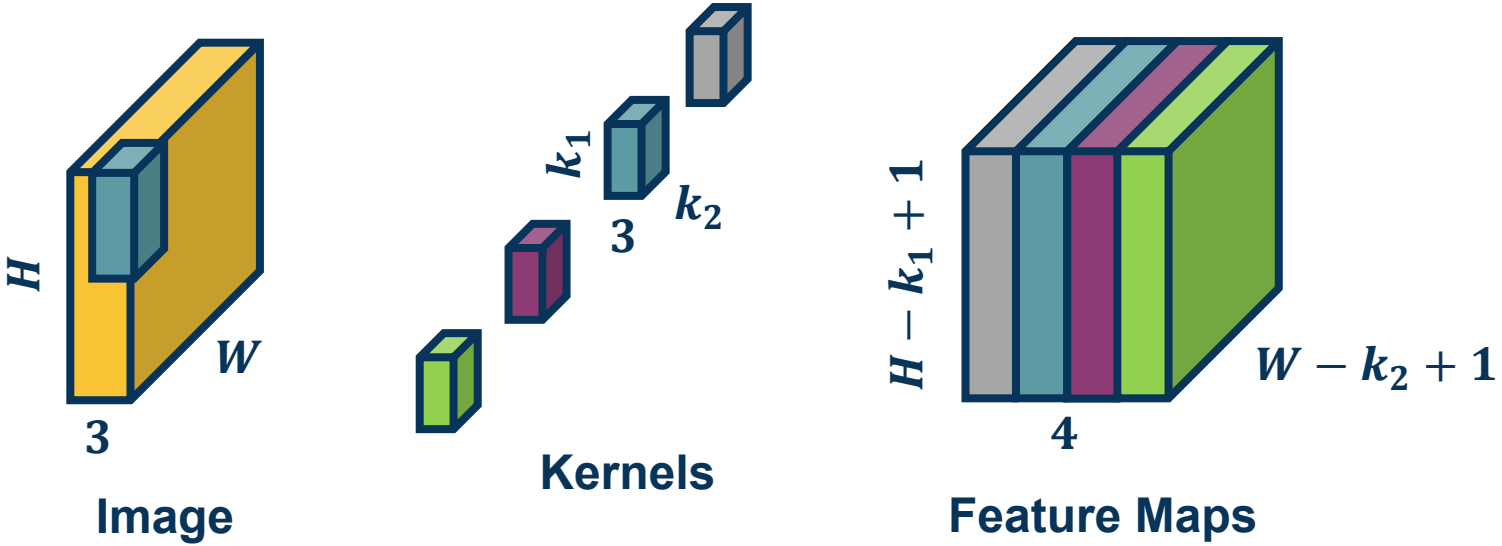
Similar to before, we perform **element-wise multiplication** between kernel and image patch, summing them up **(dot product)**

- Except with $\boldsymbol{k_1 * k_2 * 3}$ values

We can have **multiple kernels per layer**

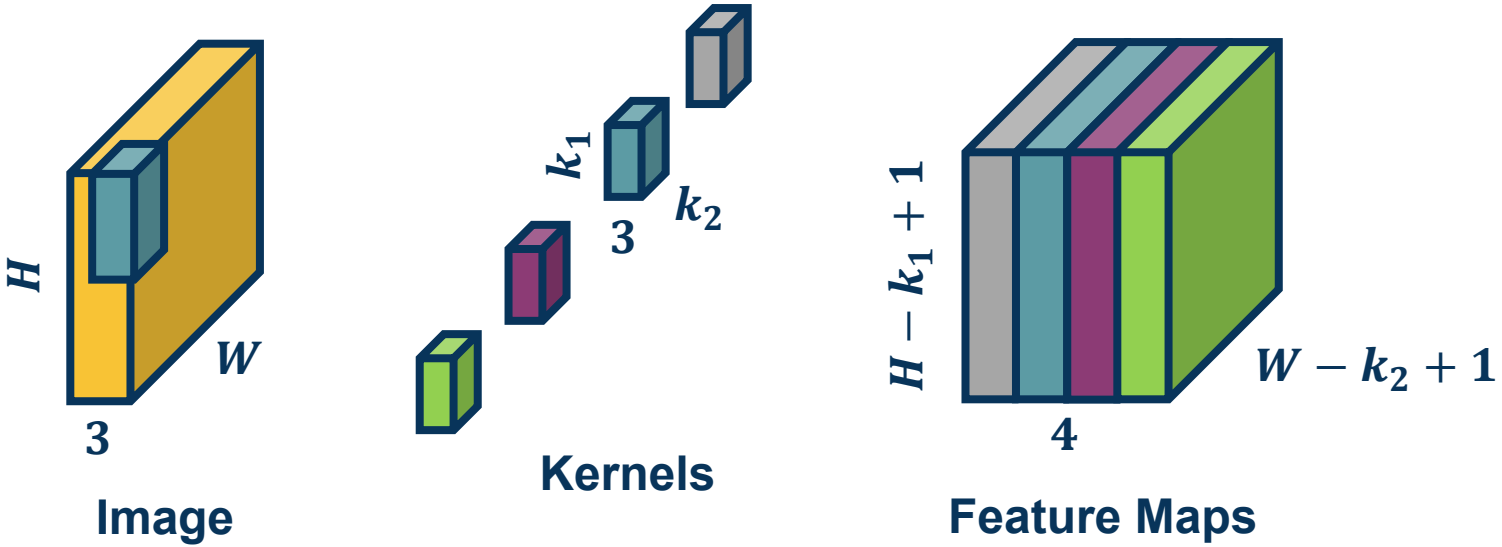
- We stack the feature maps together at the output

Number of channels in output is equal to *number of kernels*



Number of parameters with N filters is: $N * (k_1 * k_2 * 3 + 1)$

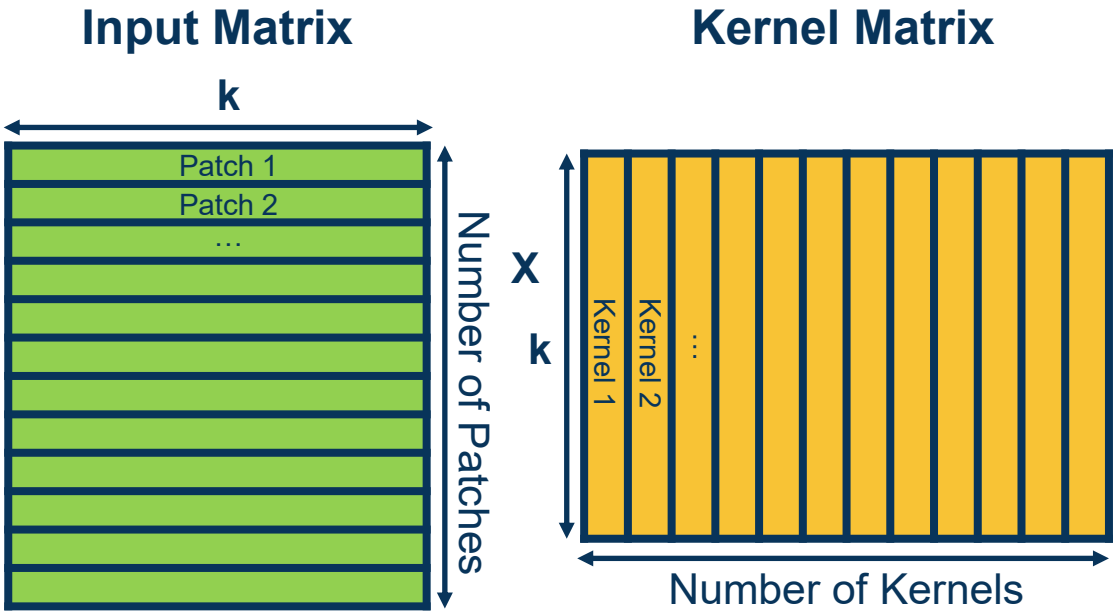
- Example:
  $k_1 = 3, k_2 = 3,\ N = 4$ *input channels* $= 3$, then $(3 * 3 * 3 + 1) * 4 = 112$



Number of Parameters

Just as before, in practice we can **vectorize** this operation

- **Step 2**: Multiple patches by kernels



Adapted from: https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/

Backwards Pass for Convolution Layer

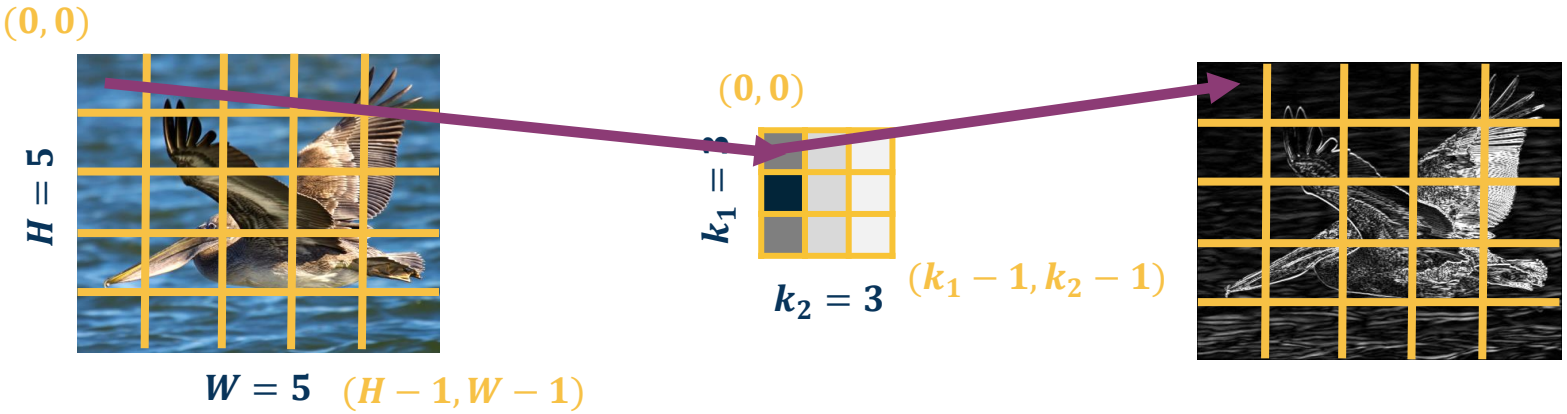
It is instructive to calculate **the backwards pass** of a convolution layer

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Similar to fully connected layer, will be **simple vectorized linear algebra operation**!
- We will see a **duality** between cross-correlation and convolution

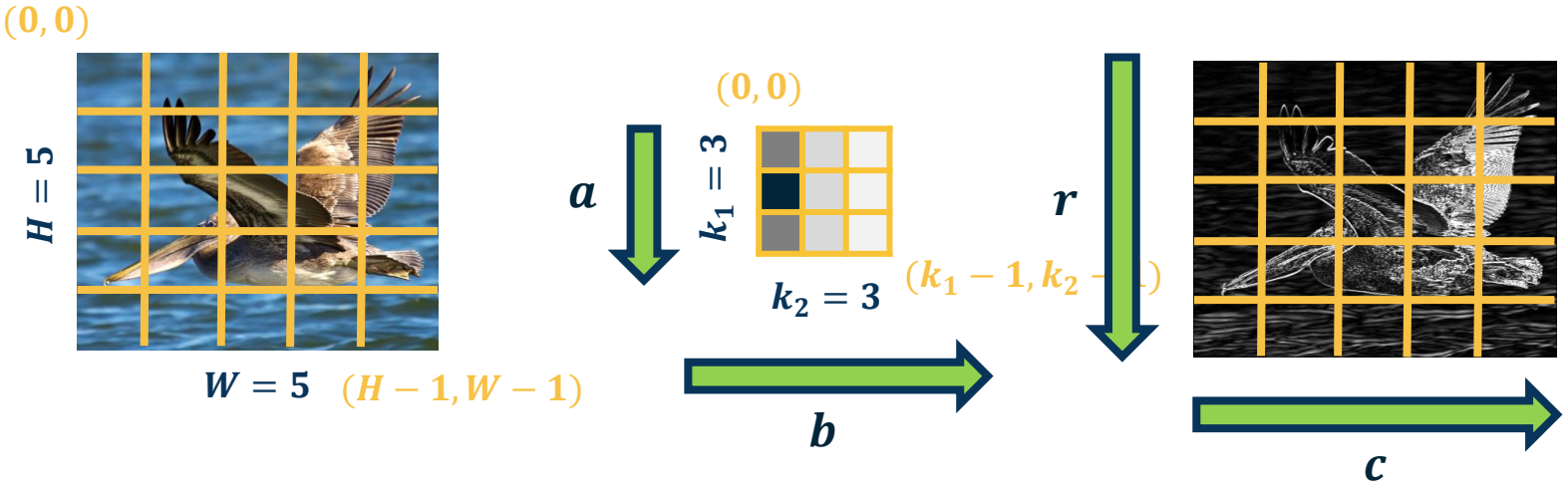
$$K' = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$



Recap: Cross-Correlation

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$



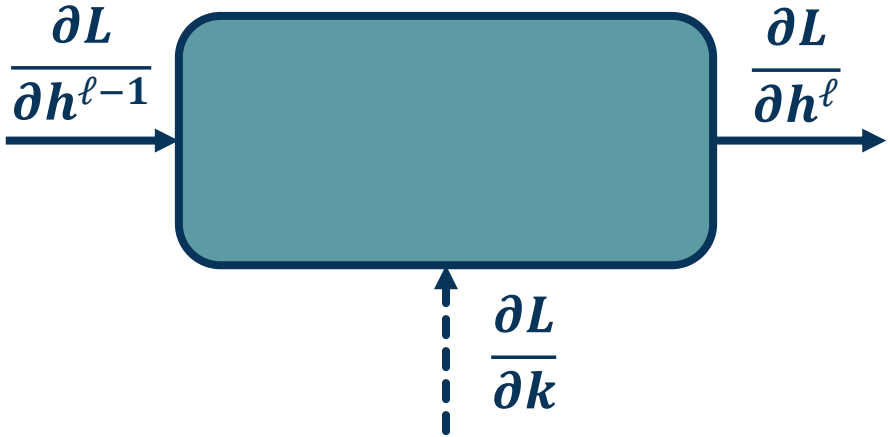
Some simplification: 1 channel input, 1 kernel (channel output), padding (here 2 pixels on right/bottom) to make output the same size

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$

$$|y| = H \times W$$

$\frac{\partial L}{\partial y}$? Assume size $H \times W$ (add padding)

$\frac{\partial L}{\partial y(r,c)}$ to access element



$$\frac{\partial L}{\partial h^{\ell-1}} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial h^{\ell-1}}$$

Gradient for passing back

$$\frac{\partial L}{\partial k} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial k}$$

Gradient for weight update

(weights = k, i.e. kernel values)

Gradient for Convolution Layer

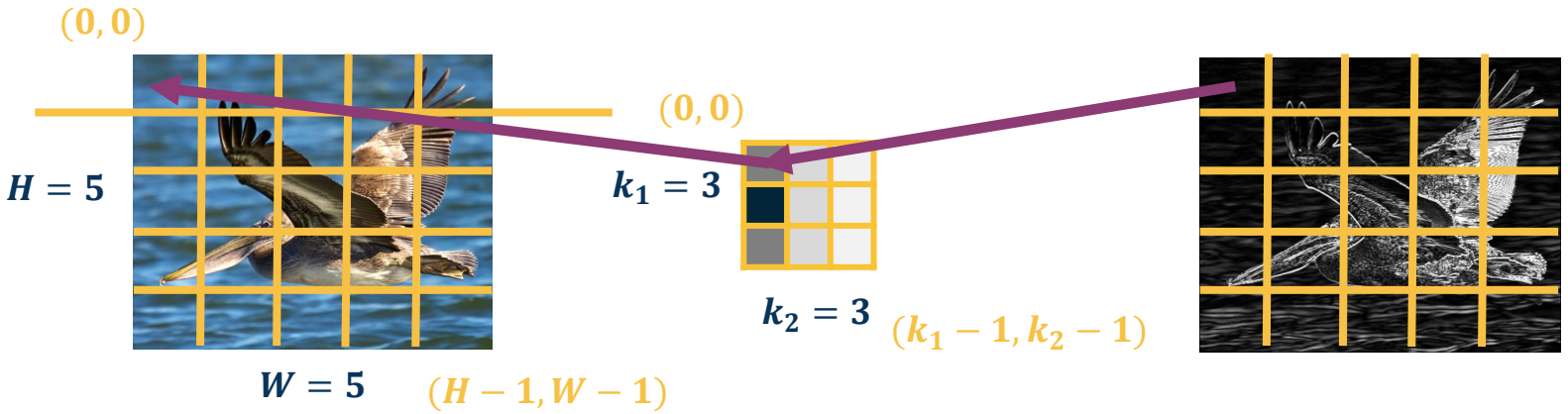
$$\frac{\partial L}{\partial k} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial k}$$

What does this weight affect at the output?

Gradient for weight update

Everything!

Calculate one pixel at a time $\frac{\partial L}{\partial k(a,b)} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial k(a,b)}$



What a Kernel Pixel Affects at Output

Georgia Tech

Need to incorporate all upstream gradients:

$$\left\{\frac{\partial L}{\partial y(0,0)}, \frac{\partial L}{\partial y(0,1)}, \cdots, \frac{\partial L}{\partial y(H,W)}\right\}$$

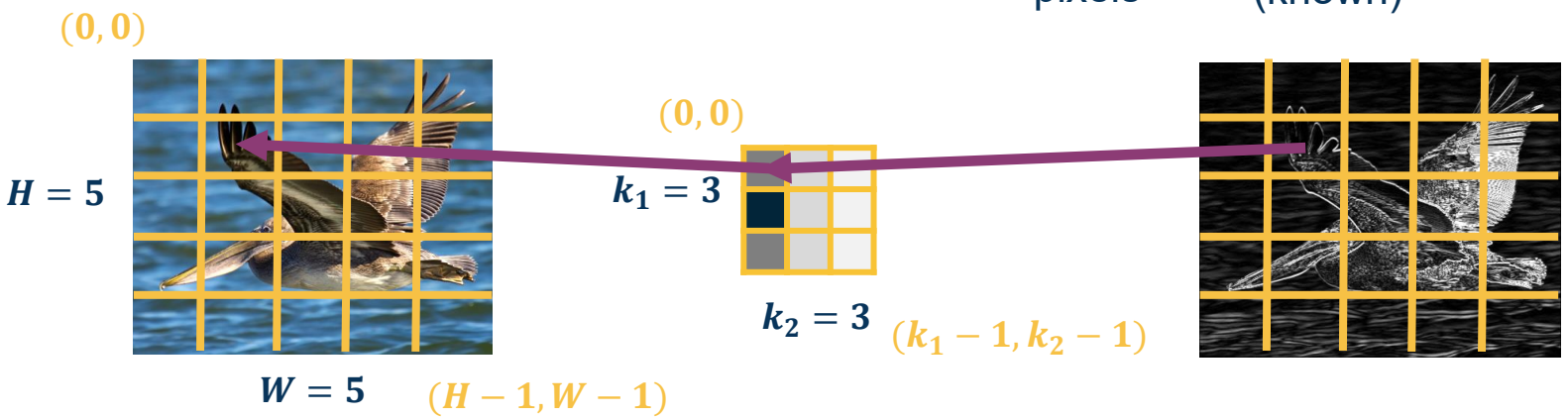
Chain Rule:

$$\frac{\partial L}{\partial k(a,b)} = \sum_{r=0}^{H-1}\sum_{c=0}^{W-1} \frac{\partial L}{\partial y(r,c)} \frac{\partial y(r,c)}{\partial k(a,b)}$$

Sum over all output pixels

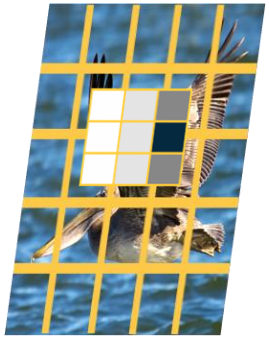
Upstream gradient (known)

We will compute

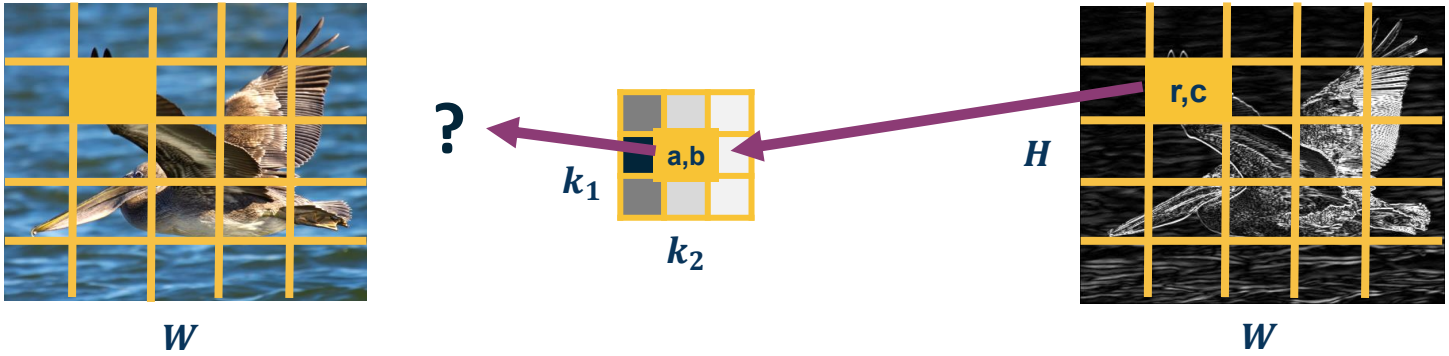


Chain Rule over all Output Pixels

$$\frac{\partial y(r,c)}{\partial k(a,b)} = ?$$



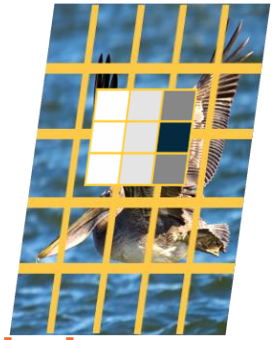
For output at y(r,c), where is placement of kernel?



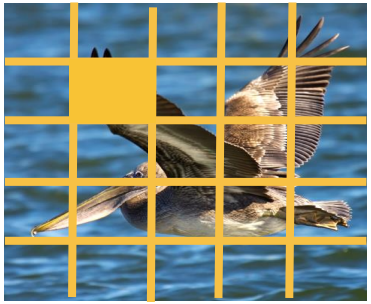
Chain Rule over all Output Pixels

Georgia Tech

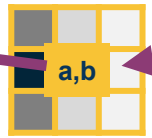
$$\frac{\partial y(r,c)}{\partial k(a,b)} = ?$$

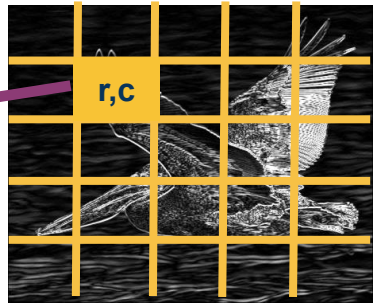


Which x pixel is multiplied by k(a,b)

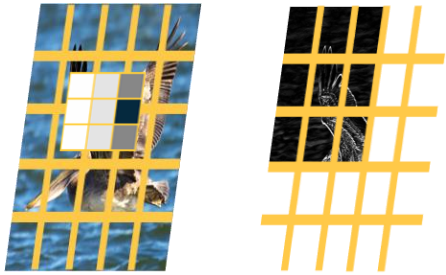


?



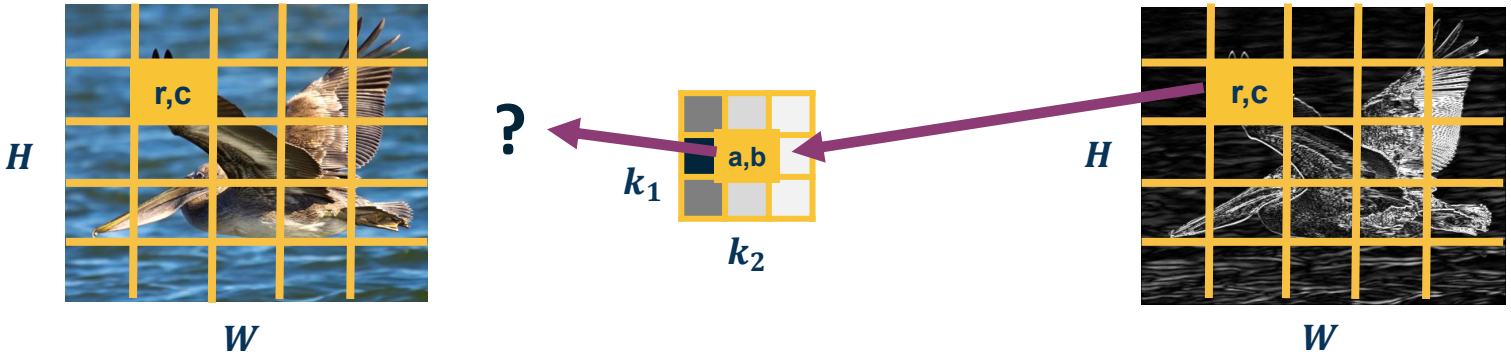


$$\frac{\partial y(r,c)}{\partial k(a,b)} = ?$$



Reasoning:

- Cross-correlation is just “dot product” of kernel and input patch (weighted sum)
- When at pixel $y(r,c)$, kernel is on input x such that $k(0,0)$ is multiplied by $x(r,c)$
- But we want derivative w.r.t. $k(a,b)$
  - $k(0,0) * x(r,c)$, $k(1,1) * x(r+1,c+1)$, $k(2,2) * x(r+2,c+2)$
  - => in general $k(a,b) * x(r+a,c+b)$
  - Just like before in fully connected layer, partial derivative w.r.t. $k(a,b)$ *only* has this term (other x terms go away because not multiplied by $k(a,b)$).



Chain Rule over all Output Pixels

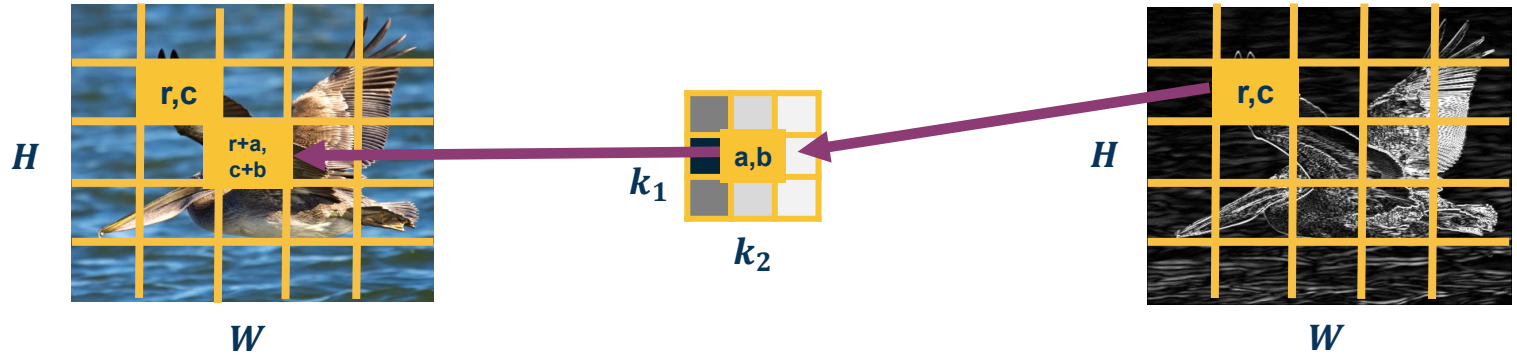
$$\frac{\partial y(r,c)}{\partial k(a,b)} = x(r+a, c+b)$$

$$\frac{\partial L}{\partial k(a,b)} = \sum_{r=0}^{H-1} \sum_{c=0}^{W-1} \frac{\partial L}{\partial y(r,c)} x(r+a, c+b)$$

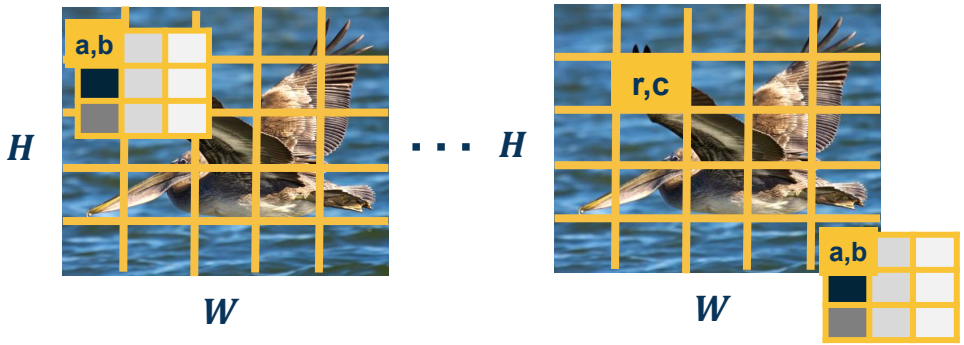
Does this look familiar?

Cross-correlation between upstream gradient and input!

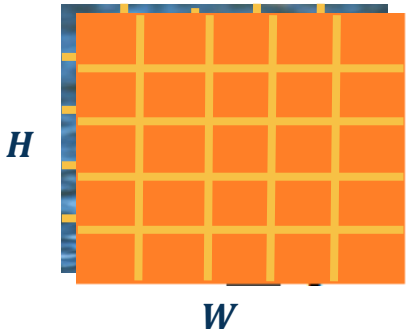
(until $k_1 \times k_2$ output)



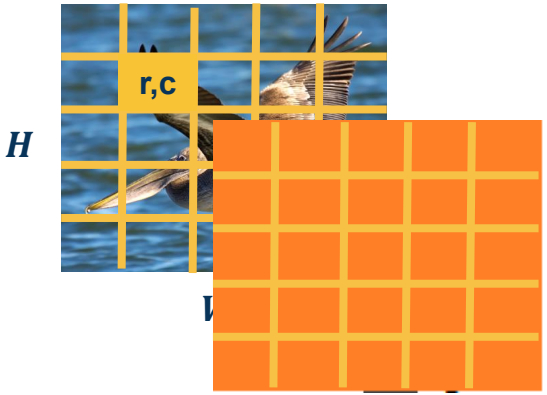
Forward Pass



Backward Pass k(0, 0)



Backward Pass $k(2, 2)$



Does this look familiar?

Cross-correlation between upstream gradient and input!

(until $k_1 \times k_2$ output)

$$\frac{\partial L}{\partial y}$$

Forward and Backward Duality

Georgia Tech

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y}\ \frac{\partial y}{\partial x}$$

Gradient for input (to pass to prior layer)

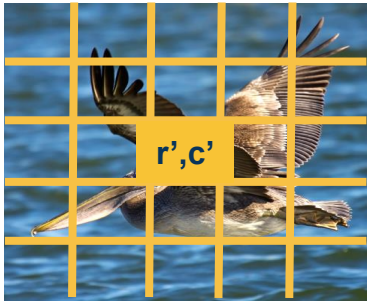
Calculate one pixel at a time $\frac{\partial L}{\partial x(r', c')}$

What does this input pixel affect at the output?

Neighborhood around it (where part of the kernel touches it)

$(0, 0)$

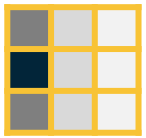
$H = 5$



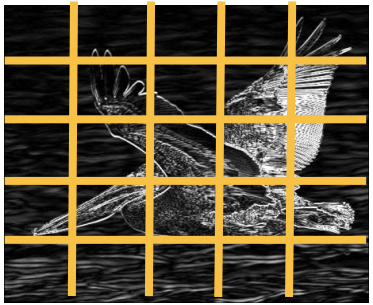
$W = 5$ $(H - 1, W - 1)$

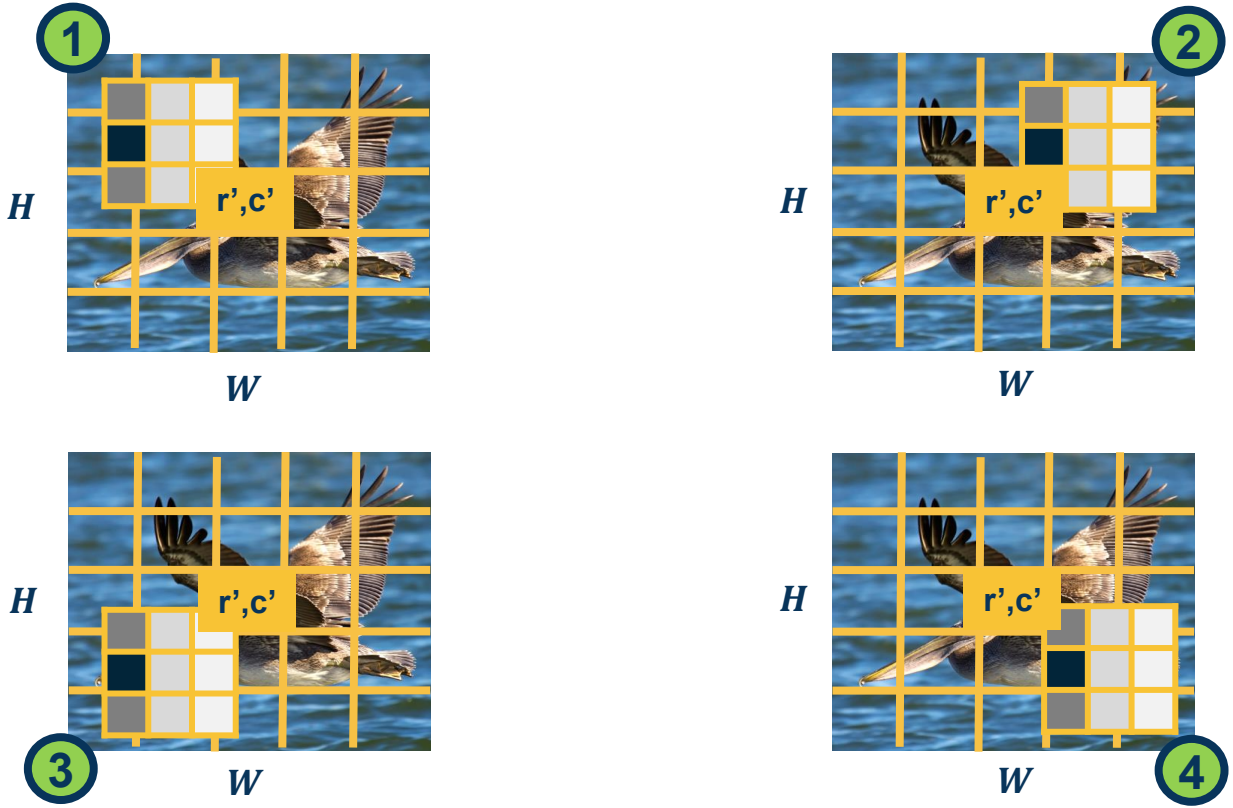
$(0, 0)$

$k_1 = 3$

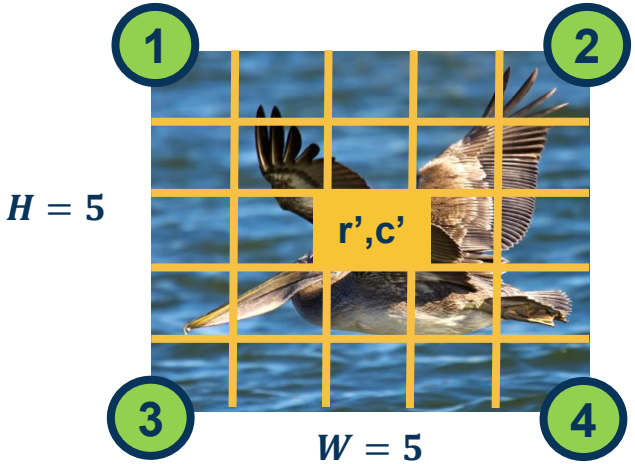


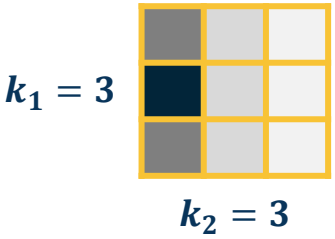
$k_2 = 3$ $(k_1 - 1, k_2 - 1)$



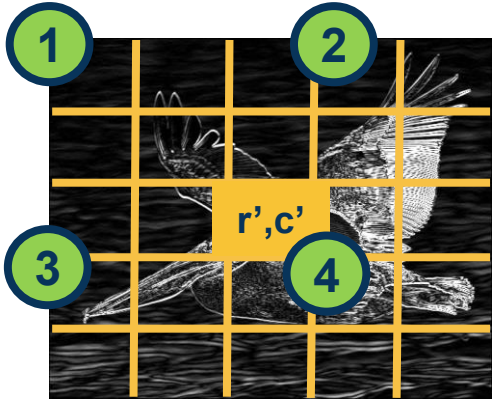


Extents of Kernel Touching the Pixel





$(r' - k_1 + 1, c' - k_2 + 1)$



This is where the corresponding locations are for the **output**

Extents at the Output

Georgia Tech

Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r', c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r', c')}$$

$$\frac{\partial L}{\partial x(r', c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(?, ?)} \frac{\partial y(?, ?)}{\partial x(r', c')}$$

Summing Gradient Contributions

Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r',c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r',c')}$$

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(?,?)} \frac{\partial y(?,?)}{\partial x(r',c')}$$

$x(r',c') * k(0,0) \Rightarrow y(r',c')$
$x(r',c') * k(1,1) \Rightarrow ?$

Summing Gradient Contributions

Georgia Tech

Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r',c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r',c')}$$

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(?,?)} \frac{\partial y(?,?)}{\partial x(r',c')}$$

$x(r',c') * k(0,0) \Rightarrow y(r',c')$
$x(r',c') * k(1,1) \Rightarrow y(r'-1,c'-1)$
…
$x(r',c') * k(a,b) \Rightarrow y(r'-a,c'-b)$

Summing Gradient Contributions

Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r',c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r',c')}$$

Let's derive it analytically this time (as opposed to visually)

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a,c'-b)} \frac{\partial y(r'-a,c'-b)}{\partial x(r',c')}$$

Summing Gradient Contributions

Definition of cross-correlation (use a', b' to distinguish from prior variables):

$$\boldsymbol{y(r', c') = (x * k)(r', c') = \sum_{a'=0}^{k_1-1} \sum_{b'=0}^{k_2-1} x(r' + a', c' + b')\, k(a', b')}$$

Plug in what we actually wanted :

$$\boldsymbol{y(r' - a, c' - b) = (x * k)(r', c') = \sum_{a'=0}^{k_1-1} \sum_{b'=0}^{k_2-1} x(r' - a + a', c' - b + b')\, k(a', b')}$$

What is $\dfrac{\partial y(r' - a, c' - b)}{\partial x(r', c')} = \mathbf{k}(a, b)$

(we want term with $x(r', c')$ in it;
this happens when $\mathbf{a}' = \mathbf{a}$ and $\mathbf{b}' = \mathbf{b}$)

Plugging in to earlier equation:

$$\frac{\partial L}{\partial x(r', c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a, c'-b)} \frac{\partial y(r'-a, c'-b)}{\partial x(r', c')}$$

$$= \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a, c'-b)} k(a, b)$$

Does this look familiar?

Convolution between upstream gradient and kernel!

(can implement by flipping kernel and cross- correlation)

Again, all operations can be implemented via matrix multiplications (same as FC layer)!

- Convolutions are mathematical descriptions of striding linear operation

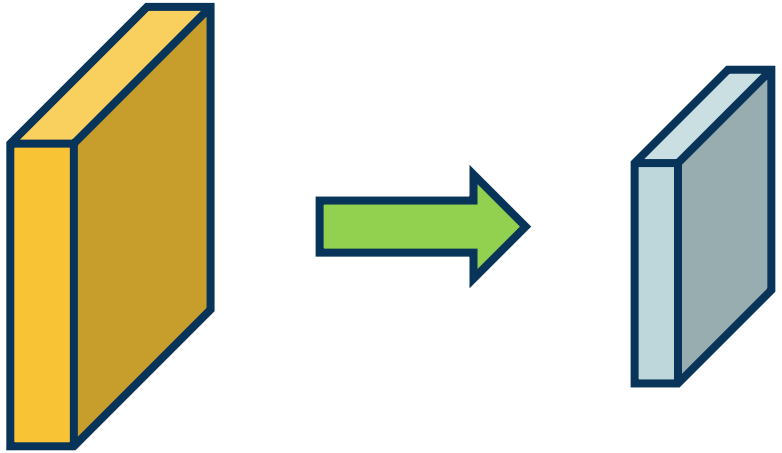
- In practice, we implement **cross-correlation neural networks!** (still called convolutional neural networks due to history)
  - Can connect to convolutions via duality (flipping kernel)
  - Convolution formulation has mathematical properties explored in ECE

- Duality for forwards and backwards:
  - **Forward**: Cross-correlation
  - **Backwards w.r.t. K**: Cross-correlation b/w upstream gradient and input
  - **Backwards w.r.t. X**: Convolution b/w upstream gradient and kernel
    - In practice implement via cross-correlation and flipped kernel

- All operations still implemented via **efficient linear algebra** (e.g. matrix-matrix multiplication)

Pooling Layers

- **Dimensionality reduction** is an important aspect of machine learning
- Can we make a layer to **explicitly down-sample** image or feature maps?
- **Yes!** We call one class of these operations **pooling** operations



Parameters

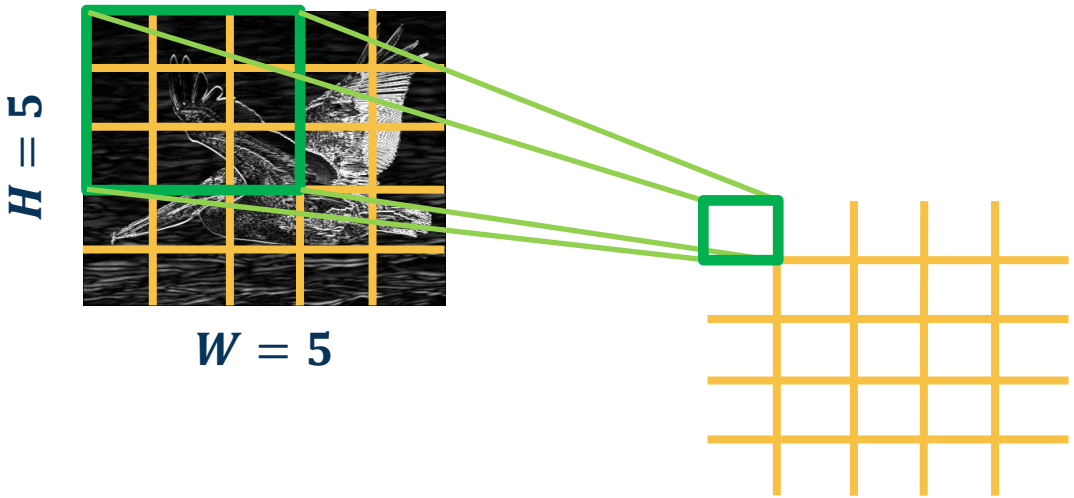
- **kernel_size** – the size of the window to take a max over
- **stride** – the stride of the window. Default value is `kernel_size`
- **padding** – implicit zero padding to be added on both sides

From: https://pytorch.org/docs/stable/generated/torch.nn.MaxPool2d.html#torch.nn.MaxPool2d

Example: Max pooling

- Stride window across image but perform per-patch **max operation**

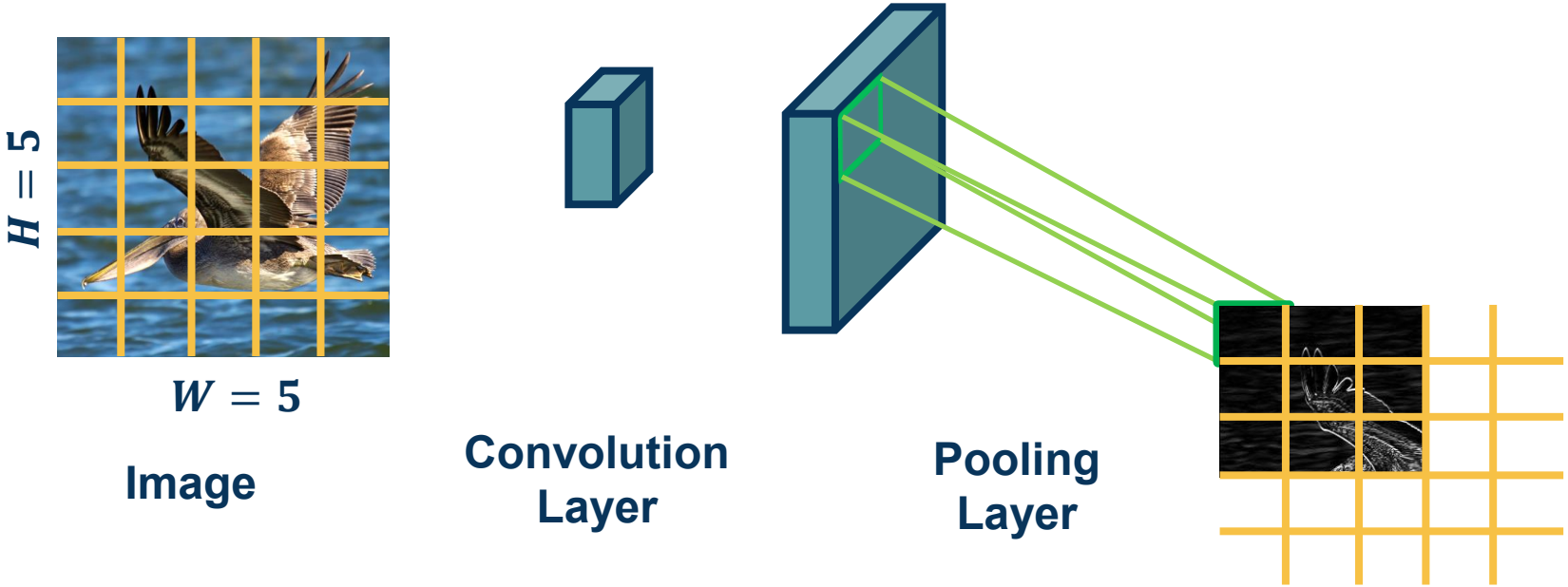
$$X(0:2, 0:2) = \begin{bmatrix} 200 & 150 & 150 \\ 100 & 50 & 100 \\ 25 & 25 & 10 \end{bmatrix} \Rightarrow \max(0:2,0:2) = 200$$



How many learned parameters does this layer have?

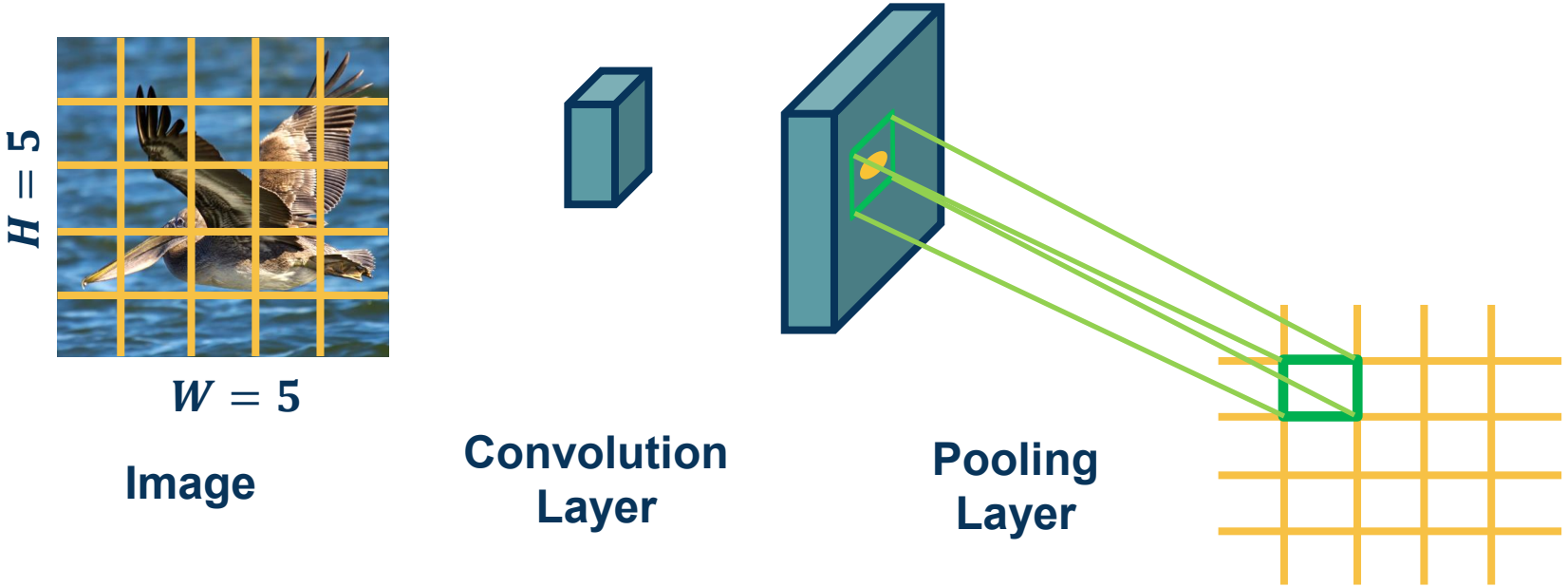
None!

Since the **output** of convolution and pooling layers are **(multi-channel) images**, we can sequence them just as any other layer



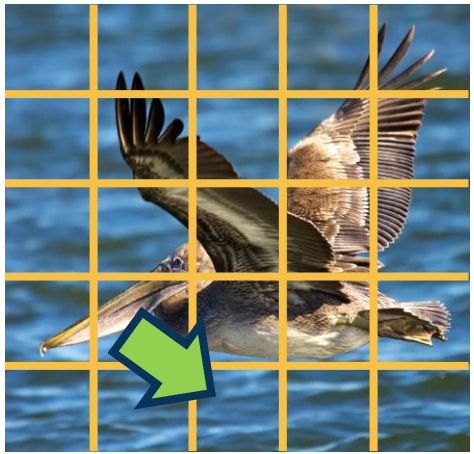
This combination adds some **invariance** to translation of the features

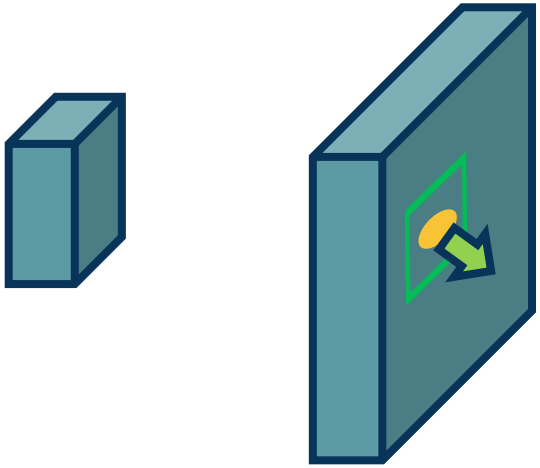
- If feature (such as beak) translated a little bit, output values still **remain the same**



Convolution by itself has the property of **equivariance**

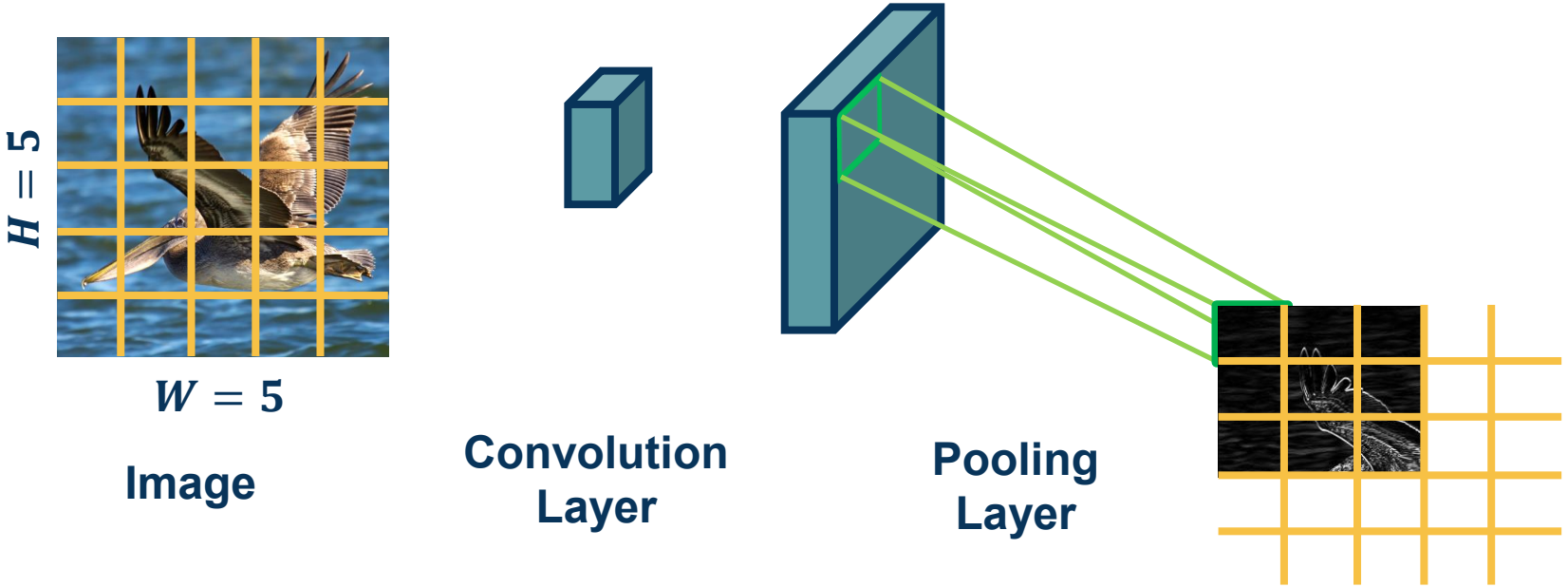
- If feature (such as beak) translated a little bit, output values **move by the same translation**



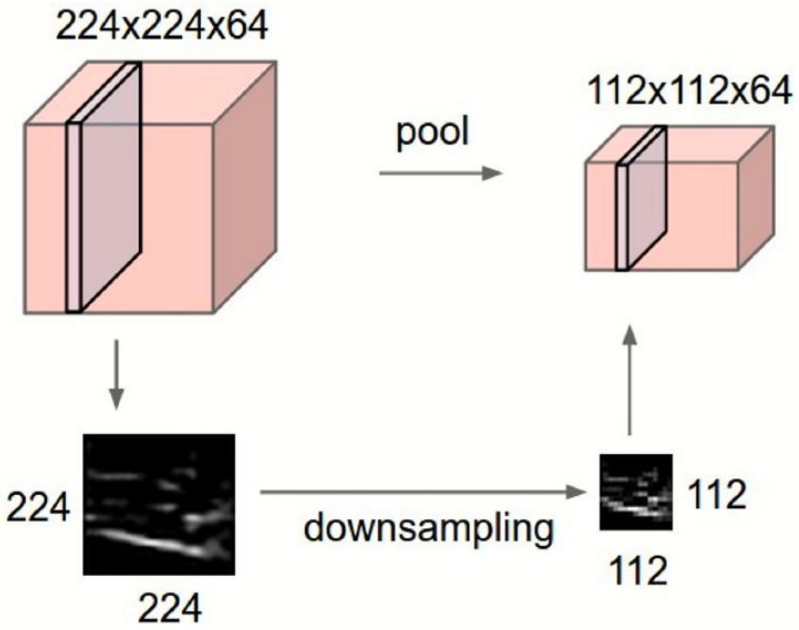


Simple Convolutional Neural Networks

Since the **output** of convolution and pooling layers are **(multi-channel) images**, we can sequence them just as any other layer



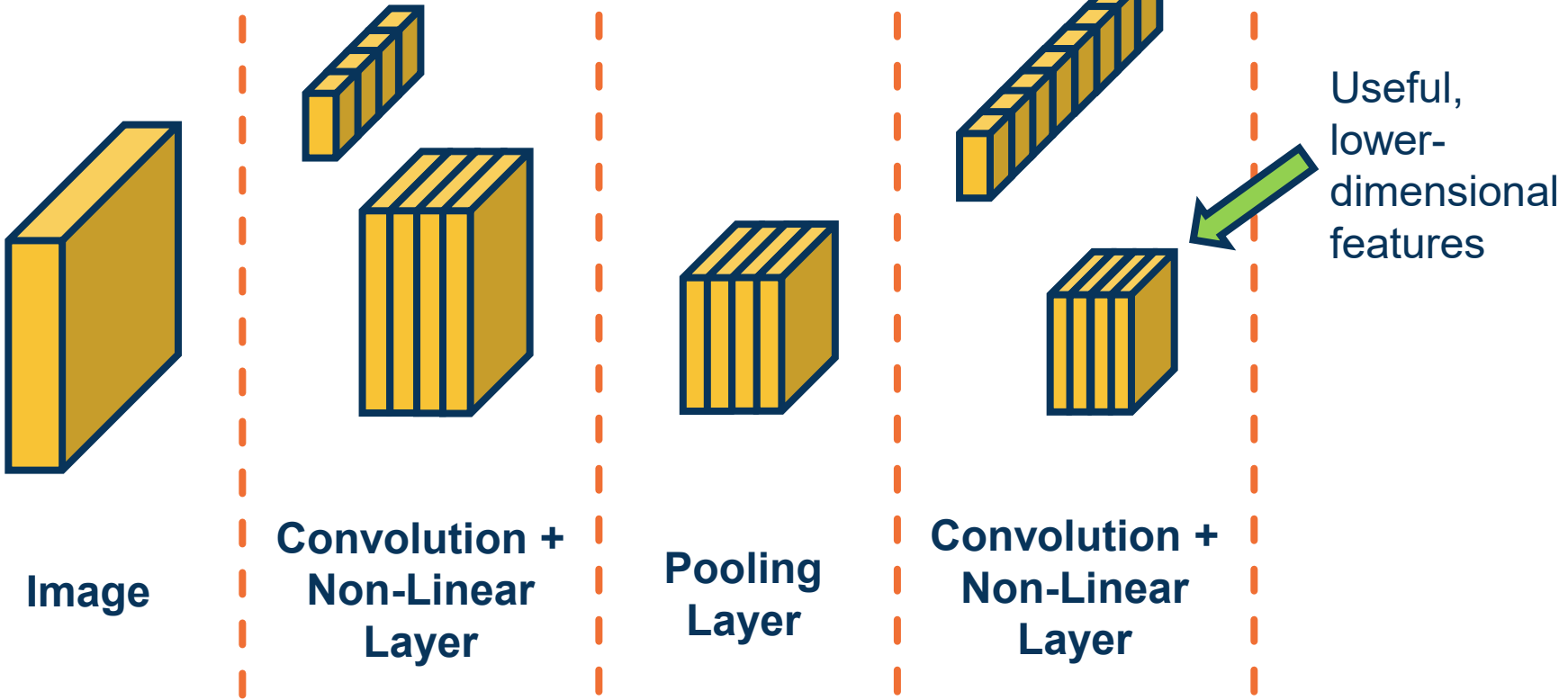
- makes the representations spatially smaller
- saves computation (GPU mem & speed), allows go deeper
- operates over each activation map independently:



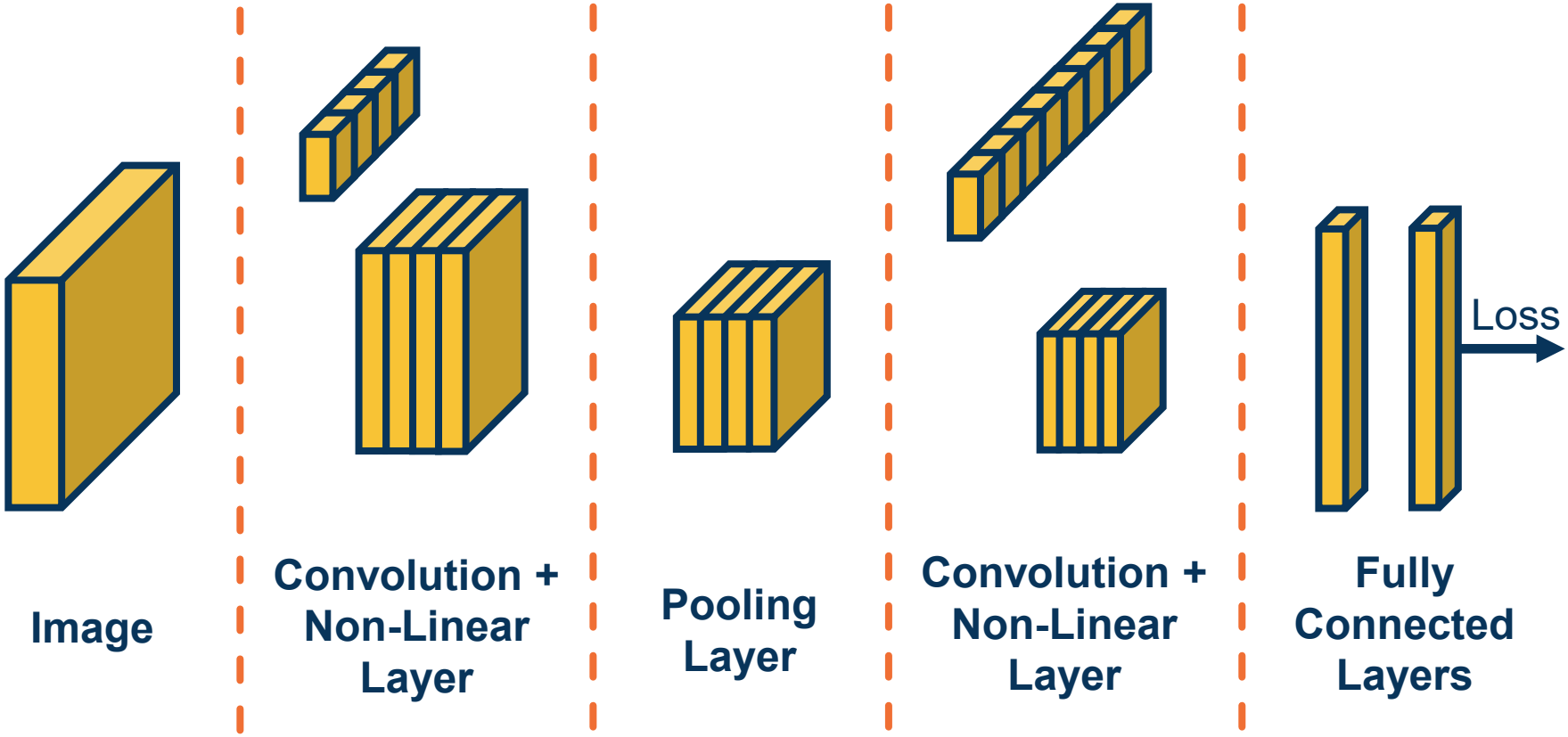
From: Slides by CS 231n, Danfei Xu

Pooling with Tensors

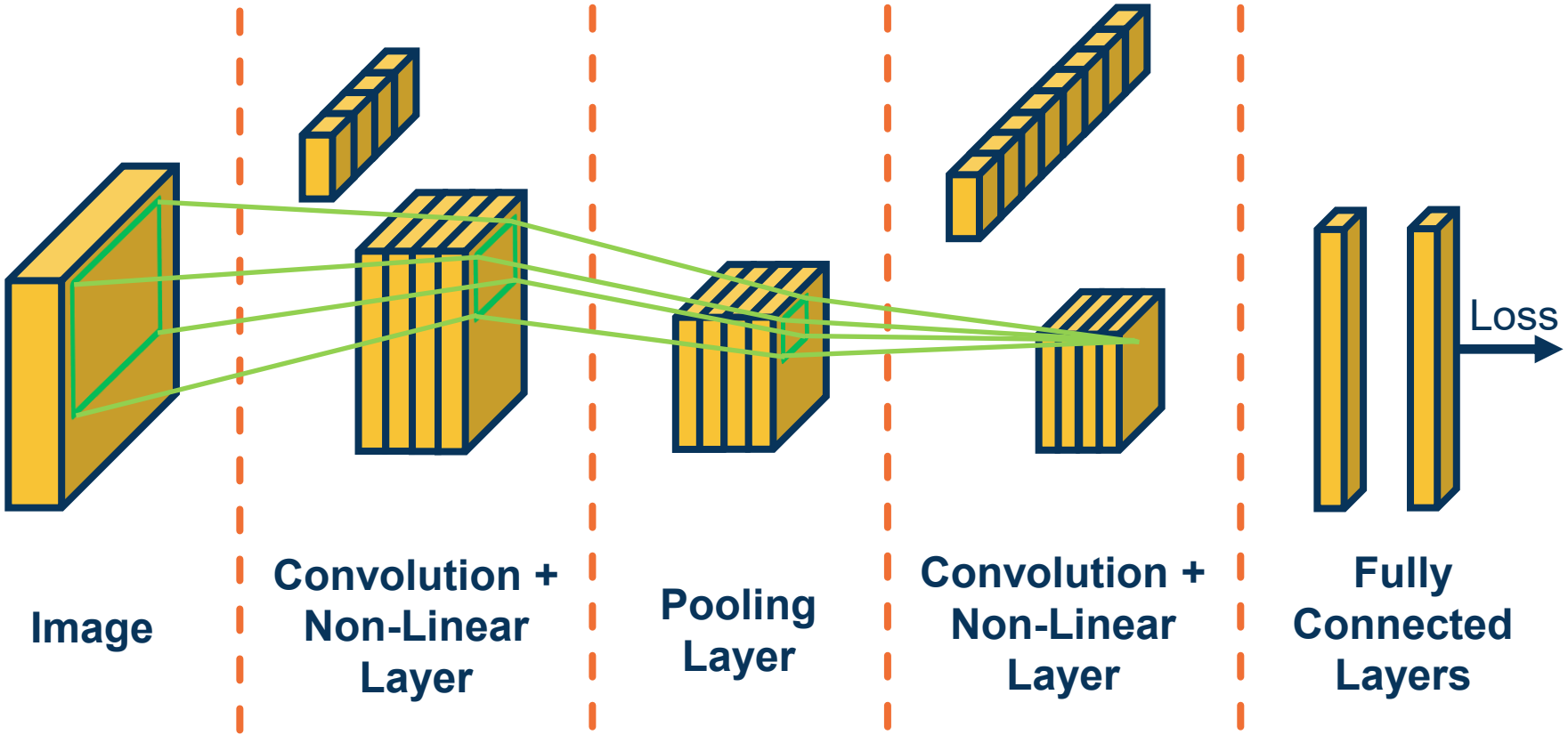
Convolutional Neural Networks (CNNs)



Alternating Convolution and Pooling



Adding a Fully Connected Layer



Receptive Fields